Calculus III
Exam 2, Summer 2003, Answers

You may use graphing calculators. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. The relation

$$\sqrt{y} + xy^2 = 34$$

determines a curve in the x-y plane. Find the slope of the line tangent to the curve at the point (2,4).

Answer. Taking differentials,

$$\frac{1}{2}y^{-1/2}dy + y^2dx + 2xydy = 0 \ .$$

Remember, to avoid unnecessary algebraic manipulations, we should evaluate at the given point as soon as we can. That time is now.So, evaluate at the point (2,4) and solve for dy/dx:

$$\frac{1}{2}(\frac{1}{2})dy + 16dx + 16dy = 0 \quad \text{so that} \quad \frac{dy}{dx} = -\frac{64}{65} \ .$$

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2. Let

$$y = \frac{x}{x^2+1}.$$

a) Find the x coordinate of the points of local maxima and minima of y.

Answer. Calculate the derivative:

$$\frac{dy}{dx} = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \ .$$

A quotient is zero when the numerator is zero, so this is zero at $x = \pm 1$. Since the derivative is positive for $-1 < x < 1$, and otherwise negative, the minimum is at $x = -1$ and the maximum at $x = 1$.
b) Find the x coordinate of the points of inflection of the curve so defined.

Answer. Now we need to calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2(-2x) - (1-x^2)2(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3} \ .$$

This is zero at $x = 0, \ \pm\sqrt{3}$. Since the sign of the second derivative changes at each point, they are all points of inflection.

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3. A monument to the mathematicians of Utah is to be built in a cylindrical form so as to enclose 6000 cu. ft. The bottom and top are to be faced with copper, costing \$30 a sq. ft., and the side surfaces of the cylinder are to be faced with stone, costing \$12 a sq. ft. What should the radius of the base of the cylinder be so as to minimize the total cost of the facing? Recall: A cylinder of base radius r and height h has volume $\pi r^2 h$, the area of the cylindrical side is $2\pi rh$, and the areas of the base and top are each πr^2.

Answer. Let V be the volume of the cylinder, and C the cost of material. We are told that $V = 6000 = \pi r^2 h$. We calculate the cost to be $C = 30(2\pi r^2) + 12(2\pi rh)$. Solve for h in terms of r: $h = 6000/(\pi r^2)$, and substitute in the formula for the cost;

$$C = 60\pi r^2 + 24\pi r(\frac{6000}{\pi r^2}) = 60\pi r^2 + 24(6000)r^{-1} \ .$$

Differentiate and set the derivative equal to zero:

$$\frac{dC}{dr} = 120\pi r - 24(6000)r^{-2} \quad \text{or} \quad C' = \pi r - 1200r^{-2} \ ,$$

which has the solution $r^3 = 1200/\pi$, or $r = (1200/\pi)^{1/3} = 7.256$ feet.

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4. Let $y = x^6 - 3x^4$. Find the intervals in which the function is increasing and decreasing, and where it is concave up and concave down.

Answer. $y' = 6x^5 - 12x^3 = 6x^3(x^2 - 2)$, $\quad y'' = 30x^4 - 36x^2 = 6x^2(5x^2 - 6)$. The function is increasing when the first derivative is positive. Since the first derivative is 0 at $x = -\sqrt{2}, 0, \sqrt{2}$, we only have to look at points in the intervals between these points. Checking at such values, we find:
increasing: $-\sqrt{2} < x < 0$ and $\sqrt{2} < x$.
decreasing: $x < -\sqrt{2}$, $0 < x < \sqrt{2}$.

The function is concave up when the second derivative is positive. Since the second derivative is 0 at $x = -\sqrt{6/5}, 0, \sqrt{6/5}$, we only have to look at points in the intervals between these points. Checking such values, we find:
concave up: $x < -\sqrt{6/5}$ and $\sqrt{6/5} < x$.
concave down: $-\sqrt{6/5} < x < \sqrt{6/5}$.

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5. Graph the function $y = x^2 + \dfrac{1}{x^2}$ showing clearly all asymptotes and local maxima and minima.

Answer. The vertical asymptote is $x = 0$, and the horizontal asymptote is $y = x^2$. Writing $y = x^2 + x^{-2}$, we easily calculate $y' = 2x - 2x^{-3} = 2x(1 - x^{-2})$, so $y' = 0$ when $x = 0, \ \pm 1$. Since the function is symmetric in the y-axis, we need only look at it for $x > 0$. Since $y' < 0$ for $0 < x < 1$ and $y' > 0$ for $x > 1$, the point (1,2) is a minimum, and $x \to +\infty$ as $x \to 0+$. Reflecting this information in the y-axis, we get that $(-1, 2)$ is also a minimum and the graph is