Calculus I
Exam 2, Summer 2002, Answers

1. A curve C in the plane is the graph of the relation $x^3 - xy^2 + y + x^2 = 2$. Find the equation of the tangent line to the curve at the point (-1,2).

Answer. Take the differential of the defining equation:

$$3x^2dx - y^2dx - 2xydy + dy + 2dx = 0 .$$

Substitute $x = -1,\ y = 2$:

$$3dx - 4dx + 4dy + dy + 2dx = 0 \quad \text{which simplifies to} \quad dx + 5dy = 0 .$$

Since dx and dy represent increments on the tangent line, replace them by $x - (-1)$ and $y - 2$ to obtain the equation of the tangent line:

$$(x+1) + 5(y-2) = 0 \quad \text{which simplifies to} \quad x + 5y = 9 \text{ or } y = (-x+9)/5 .$$

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2. A lamp is being lowered down a vertical pole at a rate of 3 ft/sec. A 6 foot man stands 20 feet away from the pole. At what rate is the shadow of the man lengthening when the lamp is 56 feet off the ground?

Answer. Draw the diagram (see below); letting x be the length of the shadow, and y the distance of the lamp from the ground. We first find the relation of x and y at any time. By similar triangles,

$$\frac{6}{y} = \frac{x}{20+x} \quad \text{which simplifies to} \quad 120 + 6x = xy .$$

Now, as the lamp falls, x and y are functions of time t. Since we want to relate the rates of change, we differentiate this expression with respect to t:

$$6\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} . \tag{1}$$

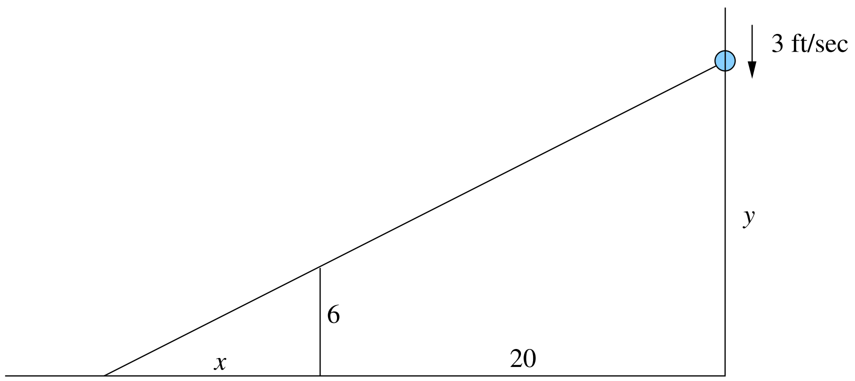
Now, when the lamp is 56 feet off the ground, the length of the shadow is found by solving

$$120 + 6x = x(56) \quad \text{giving} \quad x = 12/5 .$$

Substituting these values (and $dy/dt = -3$) into (1):

$$6\frac{dx}{dt} = \frac{12}{5}(-3) + 56\frac{dx}{dt} ,$$

which gives $dx/dt = 36/250 = .144$ ft/sec.



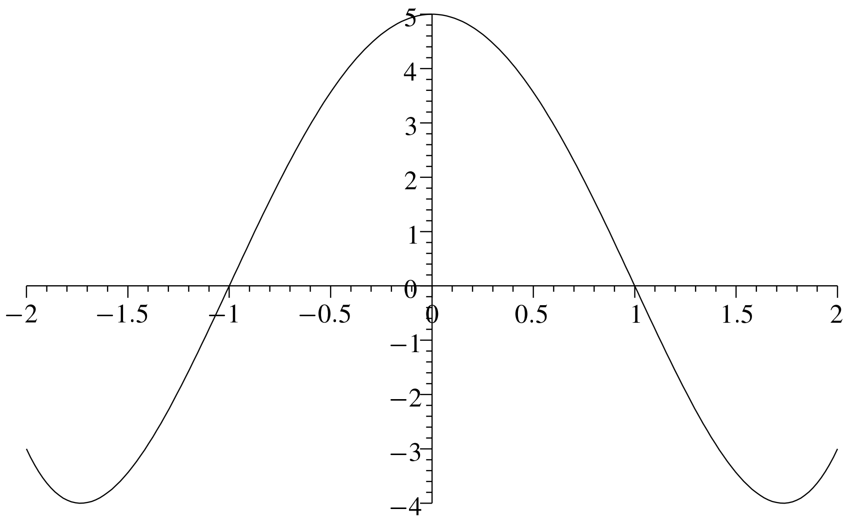
3. Let $y = (x^2-1)(x^2-5)$. For what value of x in the interval [-2,2] is y a maximum? a minimum? Find the points of inflection of the graph.

Answer. Let $f(x)$ represent the function defining y. Differentiating:

$$f'(x) = 2x(x^2-5)+(x^2-1)(2x) = 2x(2x^2-6) = 4x^3-12x \, ,$$

$$f''(x) = 12x^2-12 \, .$$

The critical points are $0, \ \pm\sqrt{3}$. We must now evaluate the function at the endpoints and critical points to find the maximum and minimum. Since the function is even, we need only check positive values of x. Since $f'(x) < 0$ in $(0,\sqrt{3})$ and $f'(x) > 0$ in $(\sqrt{3},2]$, the minimum has to be at $x = \sqrt{3}$ (and thus also at $x = -\sqrt{3}$). To find the maximum, we compare values at the endopoints and at 0. Since $y = -3$ at the endpoints, and $y = 5$ at $x = 0$, the maximum is at $x = 0$. The points of inflection are at the solutions of $f''(x) = 0$, so are at $x = \pm 1$. See the figure for a graph of the function.



4. I have to make a closed cylindrical can to hold 12 cu. ft. The material to make the top and bottom costs \$6 a sq. ft., and the material to make the side costs \$10 per sq. ft. What are the dimensions which minimize the cost? (It wil suffice to give either the radius of the base or the height).

Answer. Let r be the radius of the base, and h the height of a typical cylinder. We know that the volume must be $V = \pi r^2 h = 12$. Now the cost is \$10 for the top and bottom; that contributes $\$10(2)\pi r^2$, and \$6 for the sides, contributing $\$6(2\pi rh)$. Thus, the total cost is

$$C = 20\pi r^2 + 6(2\pi rh) = 20\pi r^2 + 144r^{-1} \, ,$$

using the equation $\pi r^2 h = 12$ to epress h in terms of r. Differentiating,

$$C = 40\pi r - 144r^{-2} \, .$$

Since we are looking for a minimum, we set this equal to 0. This gives

$$r^3 = \frac{144}{40\pi}$$

so $r = 1.046$ and $h = 3.488$.

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5. Graph

$$y = \frac{x(x-2)}{x^2-1} = \frac{x^2-2x}{x^2-1}$$

You must show enough work to explain how you found the various features of the graph.

Answer. Since the numerator and denominator have the same degree, the horizontal asymptote is $y = 1$. The vertical asymptotes are at $x = \pm 1$. Now, differentiating:

$$y' = \frac{(x^2-1)(2x-2)-(x^2-2x)(2x)}{(x^2-1)^2} = \frac{2}{(x^2-1)^2}(x^2-x+1)\ .$$

This is always positive, so the graph is always increasing. This information suffices to give the graph. See the figure.