Calculus I
Exam 2, Spring 2003, Answers

1. A curve in the first quadrant is given implicitly by the equation

$$x^2y^2 + \frac{x}{y} = 66 .$$

Find the slope of the tangent line to the curve at the point (4,2).

Answer. Rewrite the equation in exponential form: $x^2y^2 + xy^{-1} = 66$. Now take the differential:

$$2xy^2dx + 2x^2ydy + y^{-1}dx - xy^{-2}dy = 0 .$$

Evaluate at $x = 4,\ y = 2$ and solve for dy/dx:

$$2(4)(2)^2dx + 2(4)^2(2)dy + 2^{-1}dx - 4(2)^{-2}dy = 0 , \quad \text{or} \quad 32.5dx + 63dy = 0,$$

so $dy/dx = -32.5/63 = .5159$.

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2. Two variables u and v are functions of time, related by the equation

$$5u^2 + uv - 210 = 0 .$$

If $dv/dt = 3$ when $u = 6$ and $v = 5$, find du/dt.

Answer. Differentiate with respect to t:

$$10u\frac{du}{dt} + u\frac{dv}{dt} + v\frac{du}{dt} = 0 ,$$

and substitute the given values:

$$10(6)\frac{du}{dt} + 6(3) + 5\frac{du}{dt} = 0$$

giving $65(du/dt) + 18 = 0$, so $du/dt = -18/65 = -.277$.

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3. Find the minimum value of the function $f(x) = \dfrac{x^2+1}{x^2+3}$.

Answer. To find the minimum first we find all values of x at which the derivative equals zero:

$$f'(x) = \frac{(x^2+3)(2x) - (x^2+1)(2x)}{(x^2+3)^2} .$$

Setting this equal to zero, we have

$$\frac{2x(x^2+3-x^2-1)}{(x^2+3)^2} = \frac{4x}{(x^2+3)^2} = 0 .$$

The only solution is $x = 0$. Since the derivative is negative for $x < 0$ and positive for $x > 0$, this gives the minimum $y = f(0) = 1/3$.

Alternatively, one could write

$$f(x) = 1 - \frac{2}{x^2+3}$$

from which it is obvious that the minimum is 1-(2/3) = 1/3.

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4. Farmer Brown is building a rectangular chicken coop with one side against his barn. He will enclose the other three sides with chain link fence, but he will also build a chain link partition which is perpendicular to the barn. If the total area to be enclosed is 600 square feet, what should the dimensions be to minimize the amount of fence needed?

Answer. Let x be the length of the side parallel to the barn, and y the distance of that side from the barn. Then the area is $xy = 600$, and the amount of fence needed is $F = x + 3y$. Using the constraint equation, we can write F in terms of x alone as

$$F = x + \frac{3(600)}{x} = x + 1800x^{-1} .$$

Now to find the minimum value of F we differentiate and set $dF/dx = 0$:

$$\frac{dF}{dx} = 1 - 1800x^{-2} = 0$$

This give $x^2 = 1800$, so $x = 30\sqrt{2} = 42,43$, and $y = 10\sqrt{2} = 14.14$.

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5. Graph $y = \dfrac{x^2}{(x+1)(x-1)}$.
Answer. The vertical asymptotes are $x = -1$ and $x = +1$. For x between -1 and +1, y is negative except at $x = 0$ when $y = 0$. Thus, there is a local maximum at the origin, and y goes asymptotically to $-\infty$ on the insides of these lines. Writing

$$y = \frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$$

we see that y is always greater than 1 for $|x| > 1$. Thus y approaches $+\infty$ on the outsides of the asymptotes. Finally, from the second expression, we see that the horizontal asymptote is $y = 1$.

$$y = \frac{1}{1-(1/x^2)} ,$$

y approaches 1 from above as $|x| \to \infty$.