

PIGEONS, PLANETS, POINCARÉ, AND SALAD DRESSING

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The pigeon hole principle is easy to state: if you place $N + 1$ pigeons in N holes, then there must be at least one hole with at least two pigeons in it. (The proof amounts simply to counting.) One may elaborate on the principle in various ways. For instance, if $N + 1$ objects are to be placed in M holes with $M \leq N$, then at least one hole must contain at least two objects. A further elaboration goes as follows: if $kN + 1$ objects are to be placed in M holes with $M \leq N$, then at least one hole must contain at least $k + 1$ objects. (The previous case took $k = 1$. The first case took $k = 1$ and $N = M$.) And so on.

The pigeon hole principle seems trivial and in some ways it is. So it's astonishing that it can be used to solve such a wide variety of interesting problems. This week we'll focus on these kinds of problems. In practice, it is often quite easy to identify a problem as one requiring the use of the pigeon hole principle. But it is often challenging to determine what part of the problem should play the role of the "pigeons" and what should play the role of the "holes."

Many of the problems I list here were taken from Chapter 4 of *Mathematical Circles (Russian Experience)*, by Dmitri Fomin, Sergey Genkin, and Ilia Itenberg published by the American Math Society. The website <http://www.ma.umist.ac.uk/avb/Pigeon.html> is also full of good problems.

So what about the other items in this week's title? The story begins with the King of Sweden in the late 1800's. He (or rather his esteemed mathematical adviser, Mittag-Leffler) was concerned with the stability of the solar system. Once one writes down Newton's equations governing the laws of motion of the planets, it quickly becomes clear that one cannot hope that the solar system is *periodic*; that is, that one cannot hope that planets of solar system will someday return to the *exact* positions they currently occupy. Given this, one can ask: if we wait long enough, will the planets return *arbitrarily closely* to their current position? This is a version of the famous n -body problem (with $n = 10$ since I'm feeling generous and counting Pluto as a planet).

The King of Sweden offered a prize for a solution to this problem. Henri Poincaré¹ submitted a thick manuscript (of several hundred pages) giving a solution. The solution was later found to be fatally flawed, but since Poincaré introduced so many revolutionary ideas in his submission, he was still awarded the prize.

That takes care of pigeons, planets, and Poincaré. But salad dressing? It happens that one (correct) lemma proved by Poincaré in his submission to the King answers the following question. Suppose you are given a completely full bottle of oil and vinegar salad dressing. Imagine it's been sitting on a shelf for a while, so the oil is floating on top. So you need to shake it. You give it one shake, then another, then another² You happen to be dining with an inquisitive guest who asks: After a finite number of shakes, will a molecule of oil ever return to the position it originally occupied before you started shaking?

Poincaré answered this problem by employing a version of the pigeon hole principle. His answer is so useful it even has a name, the Poincaré Recurrence Theorem. The argument actually shows that almost every molecule of oil returns to its original location infinitely often. (One way to make “almost” precise is to say that those that don't return infinitely often occupy no volume in the bottle of salad dressing. But that's a topic for another day.) We'll give the salad dressing solution at the end of class today.

¹Poincaré was a remarkably influential mathematician. You might find it worthwhile to take a peek at his wikipedia entry.

²All shakes are exactly the same, no air is in the bottle, liquids are incompressible, and nothing spills out of the bottle while shaking.

EXERCISES I

1. A bag contains beads of two colors, black and white. What is the minimum number of beads that must be drawn from the bag (without looking) in order to guarantee that one has drawn two beads of the same color?
2. A million pine trees grow in a forest. It is known that no pine tree has more than 600,000 needles. Prove that there are two pine trees in the forest with the same number of needles.
3. Given any set of 12 integers, show that one may always choose two of them so that their difference is divisible by 11.
4. The maximum packing density for human hair is about 500 hairs per square centimeter, meaning that is reasonable to assume that no person has more than a million hairs on their head. The population of Utah is about 2 million. Prove that there are two people in Utah with the same number of hairs on their head.
5. Twenty-five crates of apples are delivered to a store. Each crate contains the same kind of apples, and there are three possible varieties. Prove that among these crates there are at least nine containing the same variety of apples.
6. Suppose there are M soccer teams (each with 11 team members) gathered at the Salt Lake City airport. The teams are trying to travel to San Diego for a tournament. Bad weather has canceled most flights so that now there are only 10 flights, each of which can take only M players. One team member decided to beat the weather and drove to San Diego before the storm. Prove that at least one complete team will arrive in San Diego for the tournament.
7. Given 8 different natural numbers, none greater than 15, show that at least three pairs of them have the same positive difference.

EXERCISES II

1. Consider any group of five people. Prove or disprove: there are two that have an identical number of friends within the group.

2. Several soccer teams enter a tournament in which each team plays every other team exactly once. Show that at any moment during the tournament there will be two teams which have played, up to that moment, an identical number of games.

3. What is the largest number of squares on an 8-by-8 chessboard which can be colored green so that in any arrangement of “trominos” of the form



at least one square is not colored green?

4. What is the smallest number of squares on an 8-by-8 chessboard which can be colored green so that in any tromino at least one square is colored green?

5. Ten students solved a total of 35 SAT problems. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, one who solved exactly two, and one who solved exactly three. Prove or disprove: there is also one student who solved at least five problems.

EXERCISES III

1. What is the largest number of kings which can be placed on a chessboard so that no two of them put each other in check?
2. Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.
3. Fifty-one points are scattered inside a square with a side of 1 meter. Prove that some set of three of these points can be covered by a square with side length equal to 20 centimeters.
4. 2001 flies are inside a cube of side 1. Prove that three of them are within a sphere of radius $1/11$.

EXERCISES IV

1. Prove that among any six people, there are either three people each of whom knows each other (i.e. there is a “clique” of three people), or else there are three people who are mutual strangers.
2. Show that there are two powers of two which differ by a multiple of 1987.
3. Prove that among any 52 integers, two can always be found such that the difference of their squares is divisible by 100.
4. Prove that there exists an integer divisible by 1987 whose decimal representation consists entirely of 1’s.
5. Show that there exists a number of the form $10^m - 1$ which is divisible by 2004.
6. Prove that there exists a power of three that end with the three digits 001 (in its usual base-ten form).
7. Fill each box in a 3-by-3 arrangement of boxes with either 1, 0, or -1 . For any such arrangement show that of the eight row, column, and diagonal sums, two sums must be equal.
8. The digits $1, 2, \dots, 9$ are divided in three groups. Prove that product of the numbers in one of the groups must exceed 71.
9. Consider any subset of 10 integers. Prove that there exists a subset of them whose sum is divisible by 10.
10. Fifteen boys gathered 100 nuts. Prove that two of them gathered the same number of nuts.
11. A warehouse contains 200 boots of size 9, 200 boots of size 10, and 200 boots of size 11. Of these 600 boots, 300 are left boots and 300 are right boots. Prove that there are at least 100 usable pairs of boots in the warehouse.

CHALLENGES

1. Let a be irrational. Prove that there are infinitely many rational numbers $r = p/q$ (written in lowest terms) so that

$$|a - r| < q^{-2}.$$

2. Prove that among 52 natural numbers, one can find two numbers m and n such that either the sum $m + n$ or difference $m - n$ is divisible by 100.

3. Given n integers, prove that there is a subset whose sum is divisible by n .

4. Five points are positioned inside an equilateral triangle of side 1. Prove that there are two of them a distance less than 0.5 from each other.

5. Prove that some power of 2 has a decimal expansion that begins with 1999.; i.e. prove there exists n such that

$$2^n = 1999\dots$$