DISSECTIONS OF POLYGONS AND POLYHEDRA

DAN CIUBOTARU

(1) Prove the Pythagorean theorem, by reintepreting it into a problem about dissecting two given squares and glue the resulting pieces to obtain a single square.

(2) We say that two polygons are (scissor-)equivalent if you can cut one of them into pieces and reassemble the resulting pieces to obtain the second polygon.

    Is an arbitrary triangle equivalent to a rectangle?

(3) Is a rectangle equivalent to a square?

(4) (The Bolyai-Gerwien Theorem) Any polygon is equivalent to any other polygon.

(5) Consider the statement: $P(n) =$"A square can be dissected into n squares (not necessarily equal)". For what values of $n \geq 2$ is $P(n)$ true?

(6) Similar problem with equilateral triangles in place of squares.

(7) Show that a cube can be cut into n smaller cubes for every $n \geq 55$. (perhaps the hardest case is $n = 61$.)

(8) We want to consider the analogue of the Bolyai-Gerwien theorem in 3 dimensions. The question we want to answer is if a regular tetrahedron is scissor-equivalent to a cube. Let P be a polyhedron. Max Dehn defined the following invariant of P:

$$D(P) = \sum_e \text{length}(e)\ f(e),$$

where e varies over the edges of P, and $f : \mathbb{R} \to \mathbb{R}$ is a function with the following properties:

(a) $f(0) = 0$;

(b) $f(x + \pi) = f(x)$, for all x (i.e., f is periodic with period π);

(c) $f(x + y) = f(x) + f(y)$, for all x, y (i.e., f is additive).

(Many such functions exist!) Show that if two polyhedra are equivalent, then they have the same Dehn invariant. (Hint: analyze what happens with the Dehn invariant if we cut a polyhedron P into n smaller polyhedra.)

(9) Compute the Dehn invariants of the cube and regular tetrahedron, and show that there exist choices of functions f for which the two invariants differ. In particular, the analogue of the Bolyai-Gerwien theorem cannot hold in 3 dimensions.

References

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(D. Ciubotaru) Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: `ciubo@math.utah.edu`

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