DISSECTIONS OF POLYGONS AND POLYHEDRA

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(1) Prove the Pythagorean theorem, by reinterpreting it into a problem about dissecting two given squares and glue the resulting pieces to obtain a single square.

(2) We say that two polygons are (scissor-)equivalent if you can cut one of them into pieces and reassemble the resulting pieces to obtain the second polygon.

Is an arbitrary triangle equivalent to a rectangle?

(3) Is a rectangle equivalent to a square?

(4) (The Bolyai-Gerwien Theorem) Any polygon is equivalent to any other polygon.

(5) Consider the statement: \( P(n) = "A \text{ square can be dissected into } n \text{ squares (not necessarily equal)}". \) For what values of \( n \geq 2 \) is \( P(n) \) true?

(6) Similar problem with equilateral triangles in place of squares.

(7) Show that a cube can be cut into \( n \) smaller cubes for every \( n \geq 55 \). (perhaps the hardest case is \( n = 61 \).)

(8) We want to consider the analogue of the Bolyai-Gerwien theorem in 3 dimensions. The question we want to answer is if a regular tetrahedron is scissor-equivalent to a cube. Let \( P \) be a polyhedron. Max Dehn defined the following invariant of \( P \):

\[
D(P) = \sum_{e} \text{length}(e) f(e),
\]

where \( e \) varies over the edges of \( P \), and \( f : \mathbb{R} \to \mathbb{R} \) is a function with the following properties:

(a) \( f(0) = 0 \);
(b) \( f(x + \pi) = f(x) \), for all \( x \) (i.e., \( f \) is periodic with period \( \pi \));
(c) \( f(x + y) = f(x) + f(y) \), for all \( x, y \) (i.e., \( f \) is additive).

(Many such functions exist!) Show that if two polyhedra are equivalent, then they have the same Dehn invariant. (Hint: analyze what happens with the Dehn invariant if we cut a polyhedron \( P \) into \( n \) smaller polyhedra.)

(9) Compute the Dehn invariants of the cube and regular tetrahedron, and show that there exist choices of functions \( f \) for which the two invariants differ. In particular, the analogue of the Bolyai-Gerwien theorem cannot hold in 3 dimensions.

REFERENCES


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