Suppose you are given the equations $x + y + z = a$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$, and are asked to prove that one of $x, y, \text{ and } z$ is equal to $a$. We are used to solving problems of this type by finding out where the graphs of those equations intersect—i.e. by solving for one variable in terms of the others and checking a bunch of cases.

To illustrate, try solving the above problem algebraically for $a = 2$. That is, if $x + y + z = 2$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$, show $x$ or $y$ or $z$ must equal 2.
I claim that one of $x$, $y$, and $z$ must be equal to $a$ because $a$ is a root of the polynomial $p(t) = t^3 - at^2 + bt - ab$ for any $b$. This solution is much faster, but it is not at all obvious why this observation leads to our desired conclusion. The idea is to find $b$ such that $x$, $y$, and $z$ are all the roots of $p(t)$. Then, since $a$ is also a root, $a$ must coincide with one of $x$, $y$, and $z$. But how do we find such a $b$? To investigate, we will explore the general relationship between the roots of a polynomial and its coefficients.

Definition: A polynomial in $n$ variables is homogeneous of degree $k$ if all the monomials have degrees which sum to $k$.

Examples:

1. $p(x) = x$ is a homogeneous polynomial of degree 1 in one variable.
2. $q(x, y, z) = x^4 + y^2z^2$ is a homogeneous polynomial of degree 4 in three variables.
3. $r(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ is a homogeneous polynomial of degree 2 in four variables.

We notice something special about the polynomial $r(x_1, x_2, x_3, x_4)$. If we swap $x$ and $z$ in $q(x, y, z)$, we find $q(z, y, x) = z^4 + y^2x^2$ is not the same polynomial as $q(x, y, z)$, but no matter how we reorder the $x_i$, $r$ remains the same.

Definition: A permutation is a function $\sigma$ which reorders a list of objects.

For example, let $\sigma$ be the permutation on the numbers $\{1, 2, 3, 4\}$ such that $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 3$. We see that $\sigma$ reorders the list $\{1, 2, 3, 4\}$ as $\{1, 4, 2, 3\}$. With this new notation, our observation about $r$ above can be re-stated as follows: For any permutation $\sigma$ of $\{1, 2, 3, 4\}$, we have

$$r(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = r(x_1, x_2, x_3, x_4).$$

We give the following name to polynomials with this property:

Definition: A polynomial $p$ in $n$ variables, $x_1, x_2, \ldots, x_n$ is symmetric if for any permutation $\sigma$ on $\{1, 2, \ldots, n\}$,

$$p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = p(x_1, x_2, \ldots, x_n).$$
Exercise: Is \( p(x, y, z) = x + y + z \) symmetric? Is \( p(x, y, z) = x + y \)?

There is a special collection of symmetric polynomials, called \textit{elementary symmetric polynomials}. The \( k \)th elementary symmetric polynomial in \( n \) variables, denoted \( s_k(x_1, \ldots, x_n) \), is the sum of all possible degree \( k \) monomials in \( n \) variables with each \( x_i \) appearing no more than once in each monomial. Formally, for \( k \leq n \), we will write

\[
s_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1}x_{i_2}\ldots x_{i_k}
\]

Example: \( p(x, y) = xy^2 + yx^2 \) is symmetric and homogeneous, but \textit{not} an elementary symmetric polynomial. The polynomial \( r(x_1, x_2, x_3, x_4) \) above \textit{is} an elementary symmetric polynomial.
Exercises:

1. How many monomials are there in the elementary symmetric polynomial of degree $k$ in $n$ variables?

2. List all the monomials of degree 3 in 4 variables.

3. Write down the elementary symmetric polynomials of all degrees in 3 variables.
You may have learned in algebra while learning how to factor polynomials that any integer root of a polynomial with integer coefficients will divide the degree zero term. Here is an explanation of why this should be so: Suppose $a$ and $b$ are roots of $x^2 - cx + d$. Since we know the roots, we know how factor this polynomial as $(x - a)(x - b)$. When we multiply out the factors, we see

$$x^2 - (a + b)x + ab = x^2 - cx + d;$$

consequently, $a + b = c$ and $ab = d$, so $a$ and $b$ must divide $d$.

Observe further that $a + b = s_1(a, b)$ and $ab = s_2(a, b)$, so we can rewrite the polynomial

$$x^2 - cx + d = x^2 - (a + b)x + ab = x^2 - s_1(a, b)x + s_2(a, b).$$

It happens to be true in general, that if $a_1, a_2, \ldots, a_n$ are the roots of a degree $n$ polynomial $p(x)$, of the form $p(x) = x^n + \alpha_{n-1}x^{n-1} + \ldots + \alpha_0$, then

$$p(x) = \prod_{i=1}^{n} (x - a_i) = x^n + \sum_{i=1}^{n} (-1)^i s_i(a_1, \ldots, a_n)x^{n-i}. \quad (1)$$

This implies that the coefficients $\alpha_i = (-1)^i s_i(a_1, \ldots, a_n)$. In other words, the coefficients of a polynomial can be written explicitly in terms of the roots of that polynomial using the elementary symmetric polynomials.

Example:

$$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc$$
Exercises: Compute the following polynomials in two ways—multiplying everything out manually first, then computing the coefficients via the elementary symmetric polynomials to verify they yield the same answer.

1. \((x - 1)(x - 2)(x - 3)\)

2. \((x - 1)(x + 2)(x - 3)\)
3. $(x - 2)^3(x - 3)^2$

4. $(x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$
With a little practice, you will find you can expand factored polynomials very quickly with this trick. Amaze your friends and family!

We are ready to return to our original problem. Let $b = xy + xz + yz$ and $p(t) = (t - x)(t - y)(t - z)$. Then,

$$\frac{1}{a} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{yz + xz + xy}{xyz} = \frac{b}{xyz}$$

which implies $xyz = ab$. Therefore,

$$p(t) = t^3 - (x + y + z)t^2 + (xy + xz + yz)t - (xyz) = t^3 - at^2 + bt - ab.$$ 

On the other hand, recall $p(a) = a^3 - a(a^2) + b(a) - ab = 0$, so $a$ is a root of $p$. Therefore, $a$ must equal $x$, $y$, or $z$. 


Exercises: Solve the following problems using elementary symmetric polynomials.

1. Find $a, b, c$ such that the roots of $f(x) = x^3 + ax^2 + bx + c$ are $a, b, c$. 
2. Let $a_1, a_2, a_3$ be roots of $6x^3 - 2x^2 + 3x + 5$. Find a polynomial with roots $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$. 
3. Let \( a_1, a_2, a_3 \) be roots of \( 2x^3 - 7x + 8 \). Find a polynomial with roots \( \frac{1}{a_1a_2}, \frac{1}{a_2a_3}, \frac{1}{a_1a_3} \).
4. Let $a_1, a_2, a_3$ be the three roots of $x^3 + 3x + 1$.

(a) Find a polynomial with roots $a_1^2, a_2^2, a_3^2$.

(b) Find a polynomial with roots $a_1 + a_2, a_1 + a_3, a_2 + a_3$. 
5. The Wicked Witch said that the following polynomial has 2005 integer roots: 
\( x^{2005} + 2x^{2004} + 3x^{2003} + \ldots \). Prove she is a liar. Hint: You will need the following relation:

\[
x_1^2 + x_2^2 + \ldots + x_n^2 = s_1(x_1, \ldots, x_n)^2 - 2s_2(x_1, \ldots, x_n)
\]
For the interested reader, here is the proof of equation 1. The proof is by induction on the degree of the polynomial. If our polynomial is of degree \( n = 1 \) with root \( a \), the left hand side is \( x - a \), and the right hand side is \( x - s_1(a) = x - a \), so the equation holds for \( n = 1 \). Suppose the equation holds for all polynomials of degree \( n \). Let \( p(x) \) be of degree \( n + 1 \) with roots \( a_1, \ldots, a_{n+1} \). Then, we can write

\[
p(x) = (x - a_{n+1}) \prod_{i=1}^{n} (x - a_i) = (x - a_{n+1})(x^n + \sum_{i=1}^{n} (-1)^i s_i x^{n-i})\]

where we let \( s_i \) denote \( s_i(a_1, \ldots, a_n) \) for brevity. By multiplying out the right hand side:

\[
p(x) = x^{n+1} - (s_1 + a_{n+1})x^n + \sum_{i=1}^{n-1} (-1)^{i+1}(s_{i+1} + a_{n+1}s_i)x^{n-i} + (-1)^{n+1}a_{n+1}s_n\]

Since

\[
s_1 + a_{n+1} = (a_1 + \ldots + a_n) + a_{n+1} = s_1(a_1, \ldots, a_{n+1})\]

and

\[
s_n a_{n+1} = (a_1 a_2 \ldots a_n) a_{n+1} = s_{n+1}(a_1, \ldots, a_n, a_{n+1}),\]

if we can show \( s_{i+1} + s_i a_{n+1} = s_{i+1}(a_1, \ldots, a_{n+1}) \) for all the other \( i \), we conclude the equation holds for \( n + 1 \), hence for all \( n \). By definition,

\[
s_{i+1}(a_1, \ldots, a_{n+1}) = \sum_{1 \leq j_1 < \ldots < j_{i+1} \leq n+1} a_{j_1} a_{j_2} \ldots a_{j_{i+1}}\]

By separating the sum with respect to monomials divisible by \( a_{n+1} \), we see the above is equal to

\[
\sum_{1 \leq j_1 < \ldots < j_{i+1} \leq n} a_{j_1} a_{j_2} \ldots a_{j_{i+1}} + a_{n+1} \sum_{1 \leq j_1 < \ldots < j_i \leq n} a_{j_1} a_{j_2} \ldots a_{j_i} = s_{i+1} + a_{n+1}s_i\]

so it is clear the relationship we wanted holds.