DIFFY BOXES
(ITERATIONS OF THE DUCCI FOUR NUMBER GAME)[1]

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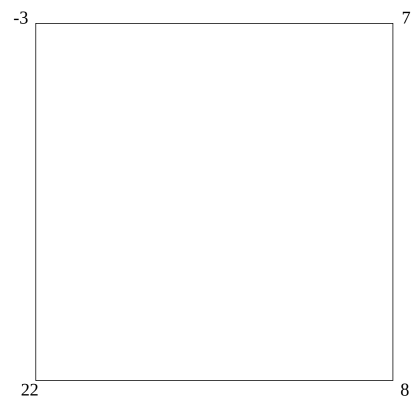
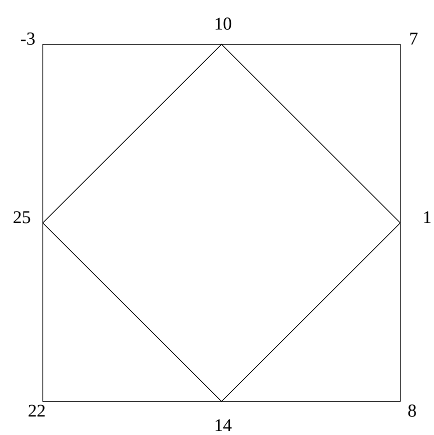
Begin at the beginning and go on till you come to the end: then stop.

— Lewis Carroll

Consider the following game. Draw a square and label each of the corners with some whole number. At the midpoint of each square write the (positive) difference between the numbers at the corresponding vertices. Then draw a new square though the midpoints and repeat the process. For instance, consider the starting square

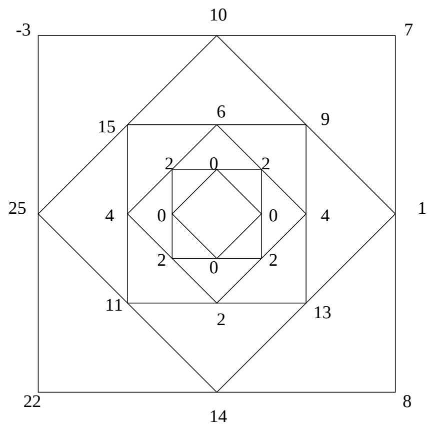


Then write in the positive differences at the midpoints and draw a new square:



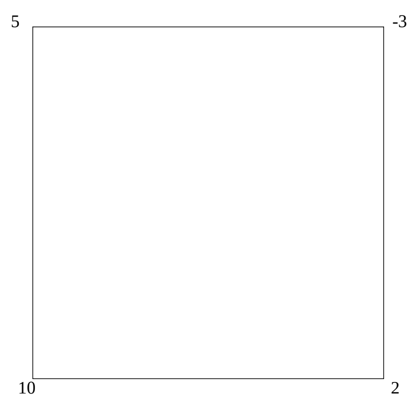
And then we repeat the process:

[1]A good reference for this material is the article "The Convergence of Difference Boxes," by A. Behn, C. Kribs-Zaleta, and V. Ponomarenko, *American Math. Monthly*, volume 112 (1995), pp. 426–438. Much of the notation and terminology I use is borrowed from that paper. An exhaustive list of references can be found at `http://mathed.uta.edu/~kribs/diffy.html`.



The remarkable observation is that we ended up at the square with all vertices labeled 0. Was that just luck? Was my choice of original square chosen in a special way so that we ended up with the zero box? What if we allow the starting numbers to be any real numbers (not just integers)? Does this change things? Given an integer k, can we find a starting box that reaches the zero box after exactly k steps? What about playing the game on pentagons? Hexagons? Cubes?

These are the main questions that we want to address over the next two weeks. Before turning to some exercises, some notation is useful. Start with a diffy box and choose any corner of it. Beginning with that corner read off the numbers at the other corners moving in a clockwise fashion. This gives a list of 4 numbers, say a, b, c, and d. Then we label the diffy box $[a\ b\ c\ d]$. For instance if we start with the box



and choose the vertex 5, then we label the box $[5\ \ -3\ 2\ 10]$. The label attached to a box is not well-defined in the following sense: if we started with the vertex 2, then we would label the box $[2\ 10\ \ 5\ \ -3]$. But this ambiguity is harmless in practice.

As an example of this notation, look at our original example. Then we can describe the sequence of diffy boxes in our new notation as

$$[-3\ 7\ 8\ 22] \to [10\ 1\ 14\ 25] \to [9\ 13\ 11\ 15] \to [4\ 2\ 4\ 6] \to [2\ 2\ 2\ 2] \to [0\ 0\ 0\ 0].$$

As a matter of terminology, we say that the original box $[-3\ 7\ 8\ 22]$ has *longevity* 5 (since it takes five moves to reach the zero box).

EXERCISES I: FIRST EXAMPLES

1. Compute the longevity of the following diffy boxes:

   (a) [0 1 0 1]

   (b) [1 0 1 0]

   (c) [0 0 1 1]

   (d) [0 1000 0 1000]

   (e) [0 0 100000 100000]

   (f) [1 2 3 4]

   (g) [1 3 2 4]

   (h) [0 1 1.5 1.8]

   (i) [0 1 1.54 1.83]

   (j) [0 1 1.543 1.839]

EXERCISES II: TRUE OR FALSE

2. Determine if the following statements are true or false. Be sure to justify your reasoning!

(a) The longevity of $[a\ b\ c\ d]$ equals the longevity of $[b\ c\ d\ a]$.

(b) The longevity of $[a\ b\ c\ d]$ equals the longevity of $[c\ b\ d\ a]$.

(c) The longevity of $[a\ b\ c\ d]$ equals the longevity of $[b\ a\ d\ c]$.

(d) The longevity of $[a\ b\ c\ d]$ equals the longevity of $[-a\ -b\ -c\ -d]$.

(e) The longevity of $[a\ b\ c\ d]$ equals the longevity of $[ar\ br\ cr\ cr]$ for any real number r.

(f) There are integers $a < b < c < d$ so that the longevity of $[a\ b\ c\ d]$ is 4.

EXERCISES III: DIFFY POLYGONS

1. We could play the same game with triangles instead of boxes. In this case, we adopt the following notation: $[a\ b\ c]$ labels a triangle whose vertices (read in clockwise fashion) are a, b, and c. We may define the longevity of a diffy triangle just as above. For instance, the diffy box [0 1 1] has *infinite* longevity since we can compute:

$$[0\ 1\ 1] \to [1\ 0\ 1] \to [1\ 1\ 0] \to [0\ 1\ 1] \to [1\ 0\ 1] \to [1\ 1\ 0] \to [0\ 1\ 1] \to \cdots .$$

On the other hand, any diffy triangle of the form $[a\ a\ a]$ has longevity 1: $[a\ a\ a] \to [0\ 0\ 0]$. Find a diffy triangle *not* of the form $[a\ a\ a]$ which has finite longevity.

2. Repeat Exercise (1) for a pentagon instead of a triangle.

Our next aim is to prove the following theorem:

Theorem. *If a, b, c and d are integers, then the diffy box $[a\ b\ c\ d]$ has finite longevity.*

The key tool in the proof of the theorem is a notion of the "size" of a diffy box:

Definition. Given a diffy box $[a\ b\ c\ d]$, define the size of $[a\ b\ c\ d]$, denoted $|[a\ b\ c\ d]|$ to be the largest difference among (not necessarily adjacent) pairs of vertices. For example, $|[-3\ 7\ 8\ 22]| = 25$ since $22 - (-3) = 25$ is the largest difference among $-3, 7, 8, 22$.

As as example, we compute the sizes of the various boxes in the first example considered above. As before, may write that sequence of boxes as

$$[-3\ 7\ 8\ 22] \to [10\ 1\ 14\ 25] \to [9\ 13\ 11\ 15] \to [4\ 2\ 4\ 6] \to [2\ 2\ 2\ 2] \to [0\ 0\ 0\ 0].$$

The corresponding sizes are easy to write down,

$$25 \to 24 \to 6 \to 4 \to 0 \to 0.$$

The key observation is that the sizes in this sequence never increase. In fact, they strictly decreases until we reached the last two terms (where the sizes were both zero). Here is the general result we need:

Lemma. *Given a pair of diffy boxes $B_1 \to B_2$, then $|B_1| \geq |B_2$. If no two adjacent corners of B_1 have the same value, then $|B_1| > |B_2|$.*

Proof. Let $a < b < c < d$ denote the four entries of B_1 arranged in increasing order. Then $|B_1| = d - a$. Meanwhile the four entries of B_2 are all between 0 and $d - a$. So the size of B_2 is at most $d - a$. This shows that $|B_1| \geq |B_2|$. When can $|B_1| = |B_2|$? That is, when does $|B_2| = d - a$? Since all entries of B_2 are between 0 and $d - a$, $|B_2| = d - a$ only if the entries 0 and $d - a$ are adjacent in B_2. But if 0 appears in B_2, then two adjacent vertices of $|B_1|$ must have the same value. This proves the second assertion of the lemma. □

Now we can return to the theorem above and try to prove it. We start with any diffy box, say B_1, and start performing our difference operation to get a sequence $B_1 \to B_2 \to B_3 \to \ldots$. At each step the size is shrinking: $B_1 \geq B_2 \geq B_3 \geq \cdots$. Let k denote the smallest integer such that $|B_k| = |B_{k+1}|$ (instead of $|B_k > B_{k+1}|$). Such a k always exists. (Why?!) If $|B_k| = 0$, then all entries of B_k are the same, B_{k+1} is the zero box, and the longevity of B_1 is k. On the other hand, suppose that $|B_k| \neq 0$. Then it looks like our argument might be stuck. To finish it we need to know one final exercise (included on the next page): any diffy box of the form $[a\ a\ x\ y]$ has finite longevity. □

EXERCISES IV: FINISHING THE PROOF.

Suppose $B = [a\ a\ x\ y]$. Show that the longevity $l(B)$ is less than or equal to 6. (This exercise completes the proof of the theorem.)

The previous theorem handles that case of *integer* diffy boxes. (Where in the proof did we use the hypothesis that all numbers involved were integers?) So that leads us to ask about noninteger diffy boxes. Consider the following table (whose first two entries you computed in the exercises above):

B	$l(B)$
[0 1 1.5 1.8]	9
[0 1 1.54 1.83]	13
[0 1 1.543 1.839]	16
[0 1 1.5437 1.8393]	22
[0 1 1.54368 1.839287]	30

At this point, the diffy boxes in the first column are really just pulled out of a hat. Our goal in the remainder of the notes is to understand a little bit about *how* they were pulled out of the hat. But before turning to that, it is impossible to resist mentioning a few more words about the numbers appearing above. The fourth entries in the diffy boxes in the above table are approaching a particular number q where

$$q = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3} \approx 1.839286755.$$

is the unique (positive) real solution to

$$x^3 - x^2 - x - 1 = 0. \tag{1}$$

Meanwhile the third entries in the diffy boxes in the above table are approaching $q(q-1)$,

$$q(q-1) \approx 1.543689013.$$

A nice coincidence[2] emerges here. Consider the Tribonnaci Sequence (which is a kind of generalization of the Fibonnaci Sequence),

$$1, 1, 1, 3, 5, 9, 17, 31, 57, \dots;$$

here the next term is obtained by adding the *three* previous terms together (and the first three terms are defined to be 1). Let T_n denote the nth term of this sequence. So $T_4 = 3$ and $T_8 = 31$, for instance. Then there is a remarkable closed formula for the nth term,

$$T_n = \text{nearest integer to} \quad 3\frac{(q+\frac{1}{3})^n \sqrt[3]{586 + 102\sqrt{33}}}{\sqrt[3]{(586 + 102\sqrt{33})^2} + 4 - 2\sqrt[3]{586 + 102\sqrt{33}}}$$

Explaining how one might arrive at a formula like this is another mystery for another day! The bizarre thing to note is the appearance of q here. At any rate, let me simply add a final line to the table above:

B	$l(B)$
$[0\ 1\ (q-1)q\ q]$	∞

Since many of you are familiar with the famous Fibonnaci sequence, let me take a moment to recall the analogous features of it. The first few terms of the Fibonnaci sequence are

$$1, 1, 2, 3, 5, 8, 12, 20, 32, \dots;$$

[2] Or maybe not a coincidence?

here the next term is obtained by adding together the previous *two* (and the first two terms are defined to be 1). If we let F_n denote the nth term of the Fibonnaci sequence, then we also have a remarkable formula

$$F_n = \text{nearest integer to } \frac{\phi^n}{\sqrt{5}},$$

where ϕ denote the Golden Mean, i.e. the unique positive real solution to

$$x^2 - x - 1 = 0.$$

(Compare with Equation (1)). So, indeed, the magic number q bears the same importance for the Tribonnaci Sequence as the Golden Mean does for the Fibonnaci Sequence. Pretty neat, huh?

But let's get back to diffy boxes. We want to have a way to visualize the longevity calculations that we have been making. In the second set of exercises above, we discovered a bunch of operations on diffy boxes that do not affect longevity. We can summarize those longevity-preserving operations as follows:

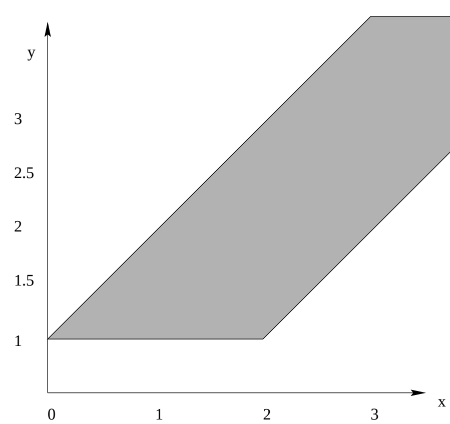
(a) translation: $l([a\ b\ c\ d]) = l([a{+}r\ b{+}r\ c{+}r\ d{+}r])$ for any real number r.
(b) negation: $l([a\ b\ c\ d]) = l([-a\ -b\ -c\ -d])$.
(c) scaling: $l([a\ b\ c\ d]) = l([ra\ rb\ rc\ rd])$ for and real number r. (So (b) is really a special case of (c).)
(d) rotation: $l([a\ b\ c\ d]) = l([b\ c\ d\ a])$.
(e) reflection: $l([a\ b\ c\ d]) = l([b\ a\ d\ c])$.

Using these operations, we now describe an algorithm to bring any diffy box into a "standard form". First we define what we mean by standard form.

Definition. A diffy box is in standard form if it is of the following three forms: [0 0 0 0]; [0 0 1 1]; or [0 1 x y] with $x \geq 0$, $y \geq 1$, and

$$x - 1 \leq y \leq x + 1.$$

In pictures, the diffy box [0 1 x y] is in standard form if (and only if) the point (x, y) lies in the following region,



Here is the algorithm to take any diffy box $[a\ b\ c\ d]$ and bring it into standard form using only the longevity preserving operations (a)–(e) described above:

. **Step 1.** Rotate (operation (d)) the box to obtain a new box $[a_1\ b_1\ c_1\ d_1]$ so that the new box has $|d_1 - a_1|$ maximal among all adjacent difference $|b_1 - a_1|$, $|c_1 - b_1|$, $|d_1 - c_1|$, and $|d_1 - a_1|$.

. **Step 2.** Observe that either a_1 or d_1 is "extreme" in the sense that either a_1 or d_1 is the largest entry in the diffy box, or else a_1 or d_1 is the smallest entry in the diffy box. Reflect (operation (e)) if necessary to obtain a new diffy box $[a_2\ b_2\ c_2\ d_2]$ so that a_2 is extreme.

. **Step 3.** If necessary, reflect to obtain a new box $[a_3\ b_3\ c_3\ d_3]$ so that $|a_3 - b_3| \geq |c_3 - d_3|$.

. **Step 4.** If a_3 is maximal, negate the box (operation (b)). Call the new box $[a_4\ b_4\ c_4\ d_4]$.

. **Step 5.** Translation (operation (a)) by a to make the first entry 0. Call the new box $[0\ b_5\ c_5\ d_5]$.

. **Step 6.** At this point, if $b_5 \neq 0$, scale (operation (c)) by $1/b$ to obtain a new box called $[0\ 1\ c_6\ d_6]$. Otherwise the box is of the form $[0\ 0\ c_6\ d_6]$. In either case, the box is now in standard form.

Here is an example. Take our original box $[-3\ 7\ 8\ 22]$. Already $22 - (-3)$ is maximal, so we don't have to do anything in Step 1. Step 2 says reflect to get the first coordinate to be maximal; so we have to reflect to get 22 in the first coordinate to be maximal. The result is $[22\ 8\ 7\ -3]$. Since $|22 - 8| \geq |7 - (-3))|$, we don't have to do anything in Step 3. Step 4 says to negate the box; we obtain $[-22\ -8\ -7\ 3]$. In Step 5, we add 33 to everything to get $[0\ 14\ 15\ 25]$. Step 6 says to divide by 14 leaving us with $[0\ 1\ 15/14\ 25/14]$ which, we observe, is indeed in standard form.

We can summarize the previous paragraph by the following list of steps:

$[a\ b\ c\ d] = [-3\ 7\ 8\ 22]$
$[a_1\ b_1\ c_1\ d_1] = [-3\ 7\ 8\ 22]$
$[a_2\ b_2\ c_2\ d_2] = [22\ 8\ 7\ -3]$
$[a_3\ b_3\ c_3\ d_3] = [22\ 8\ 7\ -3]$
$[a_4\ b_4\ c_4\ d_4] = [-22\ -8\ -7\ 3]$
$[a_5\ b_5\ c_5\ d_5] = [0\ 14\ 15\ 25]$
$[a_6\ b_6\ c_6\ d_6] = [0\ 1\ 15/14\ 25/14]$

Now try your hand at the following exercises.

EXERCISES V: STANDARD FORM

By using the above algorithm, bring the following diffy boxes into standard form. Record each step of the algorithm as in the example above

1. $abcd = [10\ 1\ 14\ 25]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

2. $abcd = [9\ 13\ 11\ 15]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

3. $abcd = [4\ 2\ 4\ 6]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

4. $abcd = [2\ 2\ 2\ 2]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

5. $abcd = [1\ 2\ 4\ 8]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

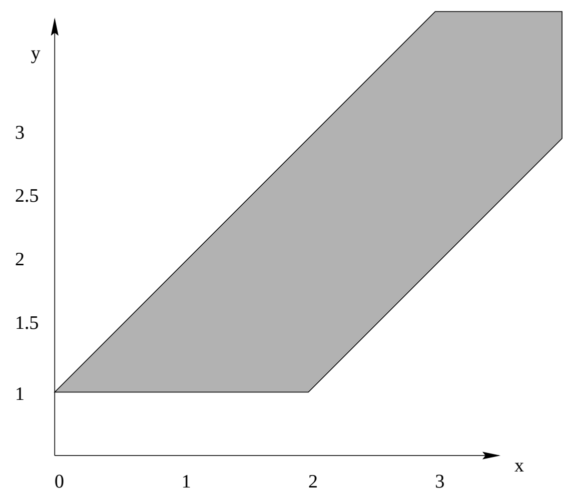
6. $abcd = [1\ 2\ 4\ 7]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

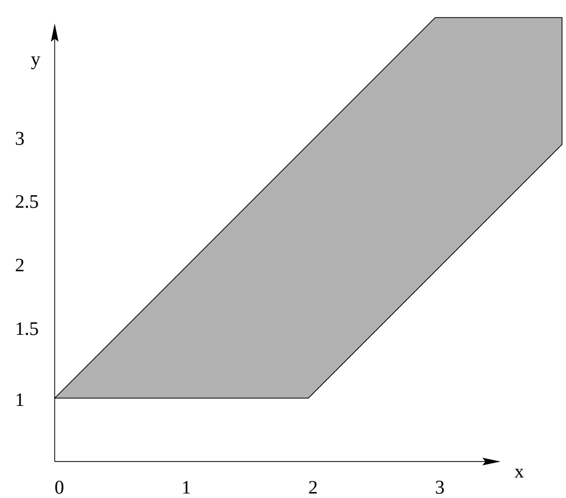
7. $abcd = [1\ 2\ 3\ 6]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

8. $abcd = [1\ 1\ 3\ 5]$
$[a_1\ b_1\ c_1\ d_1] =$
$[a_2\ b_2\ c_2\ d_2] =$
$[a_3\ b_3\ c_3\ d_3] =$
$[a_4\ b_4\ c_4\ d_4] =$
$[a_5\ b_5\ c_5\ d_5] =$
$[a_6\ b_6\ c_6\ d_6] =$

EXERCISES VI: VISUALIZING STANDARD FORM (CONTINUED)

In each of the above eight exercises, you arrived at 8 diffy boxes of the form [0 1 x y] (or [0 0 1 1]). Plot the (x, y) coordinates of the first four points on the first graph below. Plot the (x, y) coordinates of the second four points on the second graph below. Also plot the point $((q-1)q, q)$. Can you see a pattern?





We're out of time, but I can't resist the punchline. We can continue doing experiments like the ones in the last few pages of exercises: we start with a diffy box $B_1 = [0\ 1\ a\ b]$, compute the sequence of difference operations $B_1 \rightarrow B_2 \rightarrow \ldots$, covert each of these to standard form, $B_i = [0\ 1\ x_i\ y_1]$ or $[0\ 0\ x_i\ y_i]$, and then plot the sequence of points $(x_1, y_1), (x_2, y_2), \ldots$. (It's most interesting to do this for diffy boxes with long longevities, like the ones listed in the table above.) The patterns one finds are as in the examples above: the points radiate in a spiral out from a central point until one encounters the boundary. What is the central point? It's nothing but the magic box $[0\ 1\ (q-1)q\ q]$. If we start at this point, *we never move*. This doesn't mean that $[0\ 1\ (q-1)q\ q] \rightarrow [0\ 1\ (q-1)q\ q]$, but only that the difference operation applied to $[0\ 1\ (q-1)q\ q]$ has the *same* standard form. This, in turn, means that the longevity of $[0\ 1\ (q-1)q\ q]$ is infinite. Indeed it is the unique diffy box in standard form which has infinite longevity.