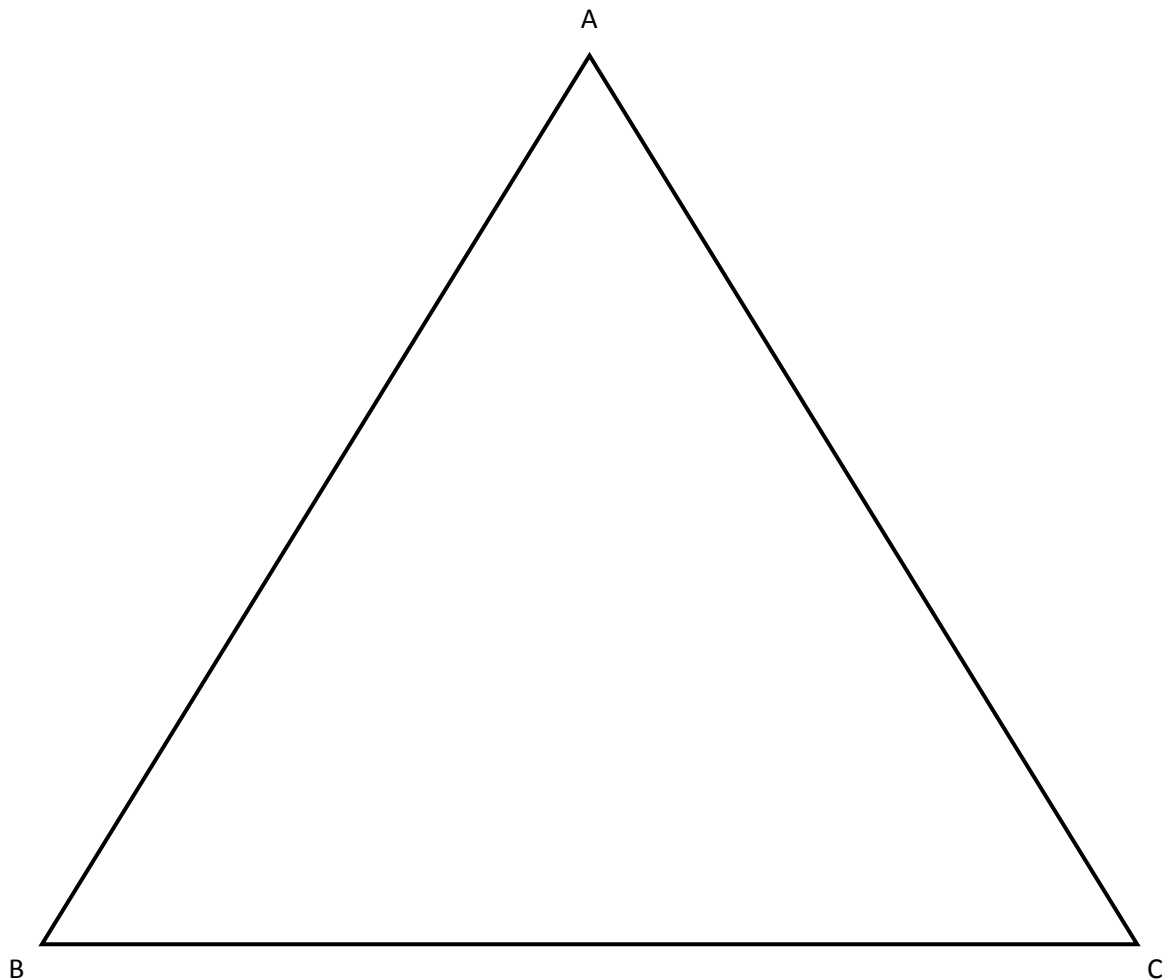


## The Chaos Game

Begin at any vertex of the triangle. Randomly determine where to go next using one of the following methods:

1. Roll one die. If you roll 1 or 2 move half the distance to vertex A. If you roll 3 or 4 move half the distance to vertex B. If you roll 5 or 6 move half the distance to vertex C.
2. Use a random number generator to generate a number,  $x$ , between 0 and 1. If  $0 \leq x \leq .3\bar{3}$  move half the distance to vertex A. If  $.3\bar{3} < x \leq .6\bar{6}$  move half the distance to vertex B. If  $.6\bar{6} < x \leq 1$  move half the distance to vertex C.

Repeat this process many times using a ruler for accuracy. Plot the points small but dark on your triangle. Do you see any patterns emerging?

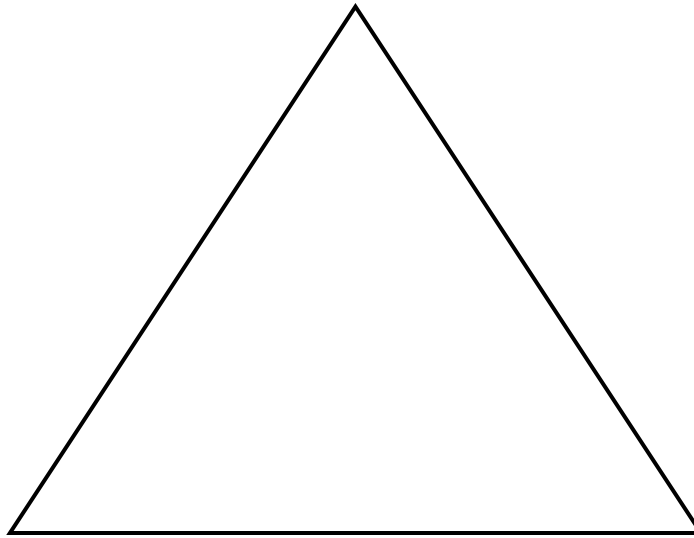


Fractals can be created through random events as we have just seen.

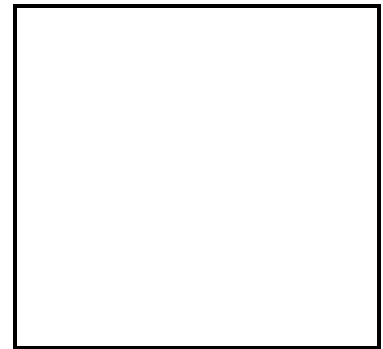
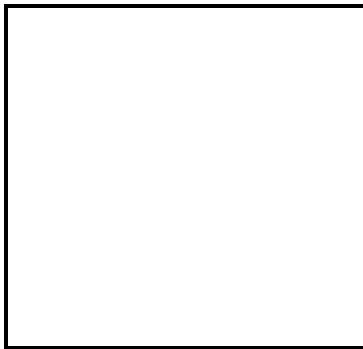
Try this game and see how you do: <http://math.bu.edu/DYSYS/applets/chaos-game.html>

## The Sierpinski Triangle and Sierpinski Carpet

Determine another way to obtain the Sierpinski Triangle that is not random and create the Sierpinski Triangle below. What is the procedure to do this?



Another example of a fractal that can be created using deterministic methods is the Sierpinski Gasket. Let's create this fractal below.

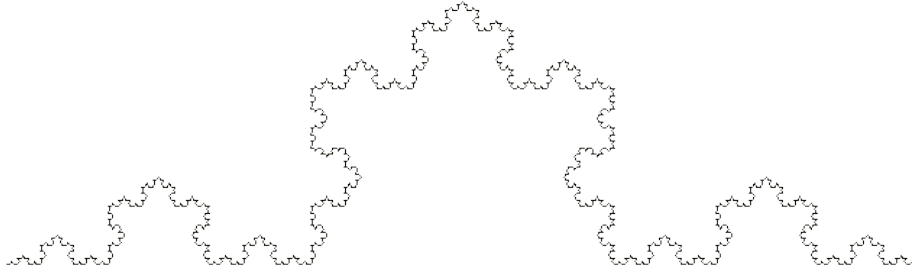


The process by which fractals are created can be thought of as **iteration**. We have a starting point and an **algorithm** (set of rules) to determine how to continue at each step. Fractals are infinitely complex as the rules can be followed infinitely many times to create much detail and self-similarity.

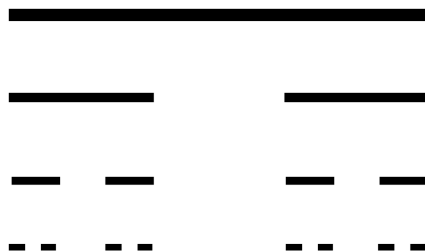
## Koch Curve and Cantor Sets

Two more examples of fractals are found below. We will create these fractals and then talk about interesting properties they have. See if you can figure out the algorithm to create them on your own and draw out the first four iterations of each.

Koch Curve



Cantor Set



## What is the length of the Koch Curve?

Iteration	length of each segment	# of segments	Total length
1	1	1	1
2	$1/3$	4	$4/3$
3			
4			
5			

What is the formula to determine length of the Koch curve for the nth iteration?

If we iterate infinitely many times what is the length of the Koch curve?

The Koch snowflake is made by starting with an equilateral triangle and iterating on all the sides. Can you figure out the final area of the Koch snowflake? What is interesting about the area of the snowflake versus the perimeter of the snowflake?

## What is the length of the Cantor Set?

Iteration	length of each segment	# of segments	Total length
1	1	1	1
2	$1/3$	2	$2/3$
3			
4			
5			

What is the formula to determine length of the Cantor Set for the nth iteration?

If we iterate infinitely many times what is the length of the Cantor Set?

Since the Cantor set ultimately has zero length we say that it has “measure zero.” That is it can be infinitely covered by intervals whose total length is arbitrarily small.

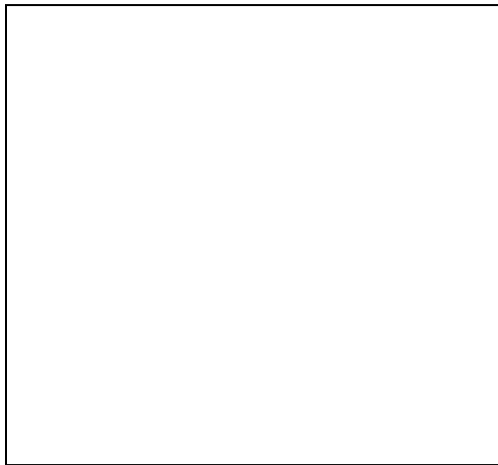
So what are the dimensions of the Cantor Set and the Koch Curve?

## Dimension

What makes something 1D, 2D, or 3D?

Will these same ideas apply to the Koch Curve or the Cantor Set? Why or why not?

To determine the dimension of fractals we will use what is known as **similarity dimension**. To figure out what this means let's consider a 2D square.

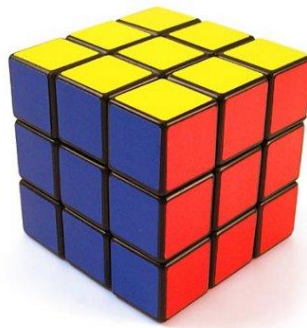


If we shrink the square by a factor of 2, how many smaller squares will it take to cover the original square?

If we shrink the square by a factor of 3, how many smaller squares will it take to cover the original square?

Do you notice any similarities between the two?

Now consider a 3D cube.



If we shrink the cube by a factor of 2, how many smaller cubes will it take to fill up the entire original cube?

If we shrink the cube by a factor of 3, how many smaller cubes will it take to fill up the entire original cube?

Noting the dimensions of the objects we just considered. Can you figure out a formula relating the scale factor ( $r$ ), the number of copies ( $m$ ), and the dimension ( $d$ )?

## Similarity Dimension

$$m = r^d$$

Solve for **d** in the above equation. This will give us a way to determine the dimension of fractals based on the properties of their self-similarity.

Use this formula to determine the dimension of the following fractals.

Fractal	Scale factor, r	Number of copies, m	Dimension, d
Koch Curve			
Cantor Set			
Sierpinski Triangle			
Sierpinski Carpet			
Box Fractal			
Detour Fractal			

A key feature of fractals is that their dimensions are not integer values.

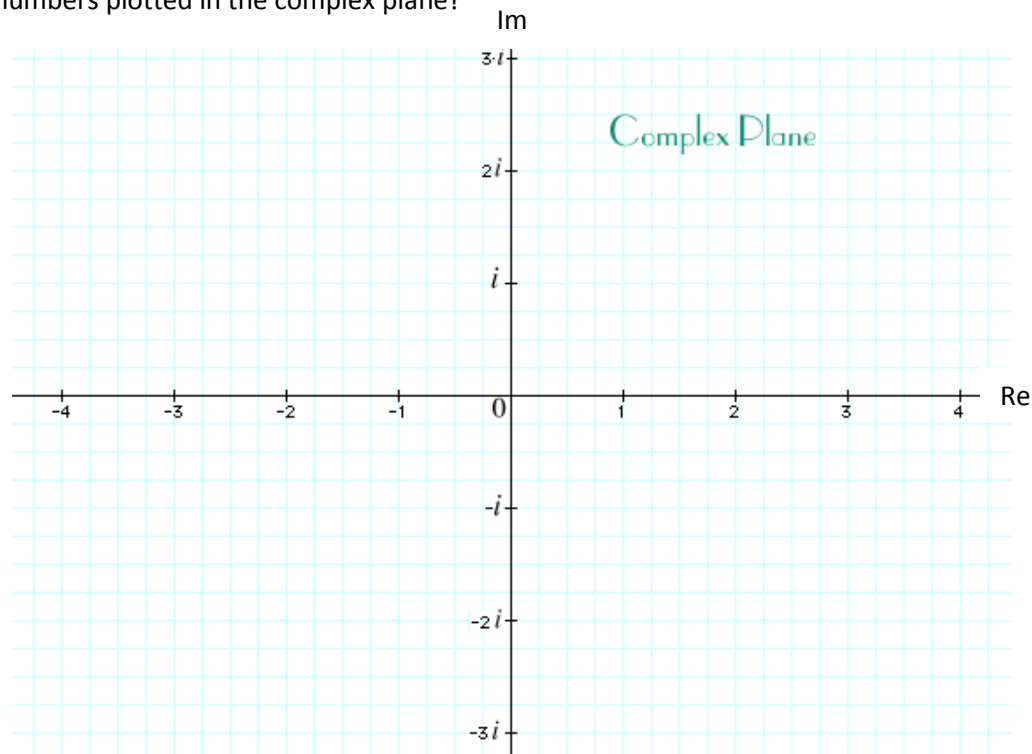
Try starting with a straight line and creating your own fractal which is self-similar. What is the dimension of the fractal you created?

## Complex Function Iteration and the Mandelbrot Set

To discuss the most well-known fractal we must review complex numbers and complex iteration.

What does  $i$  represent?

How are numbers plotted in the complex plane?



Plot the following numbers:  $3 + 2i$ ,  $.5 - i$ ,  $4$ , and  $-2i$  just for practice.

Iterating a function is much like the iteration we've already done today. We have a starting value and an algorithm for the steps to follow to complete each iteration. To create the famous Mandelbrot set we will iterate on the function

$$f(z) = z^2 + c$$

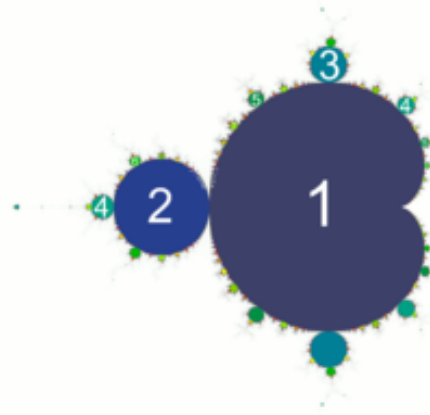
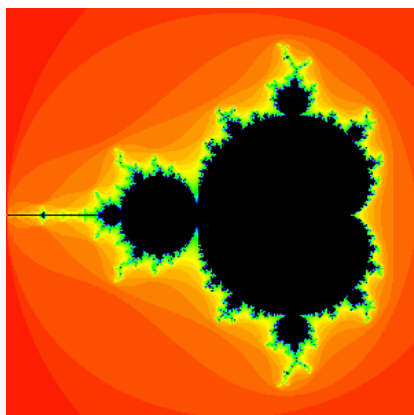
We always use our starting value as  $z = 0$ . The point  $c$  is in the Mandelbrot set if the iterates do not diverge to infinity, i.e. the values of the iterates stay close to the origin and don't ever get "too big."

Use complex function iteration as follows to find the first 10 iterates for each value of  $c$ . Then determine if the point  $c$  is in the Mandelbrot set.

$$\begin{aligned}
 z_0 &= 0 \\
 z_1 &= z_0^2 + c = c \\
 z_2 &= z_1^2 + c = c^2 + c \\
 z_3 &= z_2^2 + c \\
 &\vdots \\
 z_{n+1} &= z_n^2 + c
 \end{aligned}$$

c value	$z_1$	$z_2$	$z_3$	$z_4$	$z_5$	$z_6$	$z_7$	$z_8$	$z_9$	$z_{10}$	M set? Y/N
.25											
1											
-1											
-1.3 +.01i											
i											
$2 - 3i$											

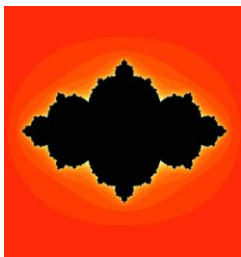
Do you notice any patterns arising with the iterates? Within the Mandelbrot set there are fixed points, periodic points, eventually fixed or periodic points, and other types of orbits. This simple function gives rise to interesting behavior and creates a very detailed fractal.



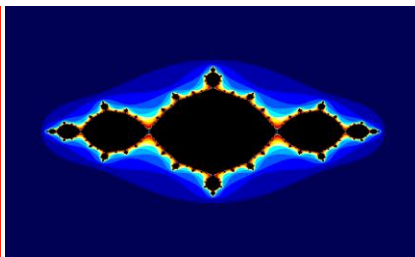


## Julia Sets

Again consider the function  $f(z) = z^2 + c$ . Now fix the value of  $c$  and see what happens as we allow  $z$  to vary. Each value of  $c$  in the complex plane has an associated Julia set. The **Filled Julia set** is defined as all the “prisoner” values of  $z$ , i.e. all the values of  $z$  for which the iterates remain bounded (similar to what we looked at in determining the Mandelbrot Set). Everything outside the filled Julia set is part of the “escape” set as the iterates escape to infinity. The Julia Set is defined as the points that make up the boundary of the escape set. Therefore, if you move even the slightest bit outside the Filled Julia set, the iterates escape to infinity. This behavior shows that the system is highly sensitive to initial conditions: one slight change in the initial value produces a very large change in the outcome. This is the essence of Chaos Theory (Butterfly effect).



$c = 1 - \text{golden ratio}$



$c = -1$



$c = -.11 - .67i$

The values of  $c$  which have **connected** Julia sets make up the points in the Mandelbrot Set.

