Intro to Number Theory: Hints

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Preliminaries

Base 10 Arithmetic

Problems

- What is $7777 + 1$ in base 8?

  Hint: In base 10, $7 + 1 = 8$, but in base 7, $7 + 1 = 10$.

- In what base is 21^2 equal to 225_{10} ?

  Hint: call the base b. Then in base 10, $(2 * b + 1)^2 = 225$.

- You ask your cyborg friend what it would like to eat. It replies "48,879". Knowing that your cyborg friend thinks in hexidecimal but speaks in decimal, what should you feed it?

  Hint: It's first useful to compute some powers of 16; $16^2 = 256$, $16^3 = 4096$, and $16^4 = 69632$. Notice that this last power of 16 is larger than the given number, so we'll only need 4 hexidecimal digits.

Fundamental Theorem of Arithmetic

Problems

- Factor 120 uniquely into primes.

  Hint: 120=5!.

- Three integers (x, y, z) satisfy $34x + 51y = 6z$. If y and z are primes, what are these numbers?

  Hint: Writing $17(2x + 3y) = 6z$ shows that z is divisible by 17.

- Prove that $\sqrt{p}$ is an irrational number for any prime p.

  Hint: Suppose that $\sqrt{p}$ is a rational number. Then there exist two integers, n and m with no common divisor such that $\sqrt{p} = n/m$.

- Suppose that p is the largest prime number. Is $p! + 1$ divisible by any primes $\leq p$? Is this a contradiction?

  Hint: The number $p!$ is divisible by all primes $\leq p$. Can you see why?

Divisibility Tests

Divisibility by Powers of 2

Problems

- Is $1,234,567,890$ divisible by 2?

  Hint: Look at the last digit.

- Is $121^{13} - 101^4$ divisible by 2?

  Hint: Any number ending in 1, when raised to any power, still ends in 1. Can you see why?

- Prove that $1782^{12} + 1841^{12} \neq 1922^{12}$. Do you know why your calculator is wrong?

  Hint: 1782^2 and 1922^2 are each divisible by 2.

- How do you prove the 2^n case?

  Hint: Notice that 100 is divisible by 4, that 1000 is divisible by 8 and that in general 10^n is divisible by 2^n.

Divisibility by 3 and 9

Problems

- Does the above proof also work for the case of divisibility by 9?

  Hint: Are the terms that were described as being divisible by 3 actually divisible by 9 as well?

- Is $1,234,567,890$ divisible by 3?

  Hint: $1+2+3+4+5+6+7+8+9+0 = 45$.

- Is $326^2 - 325^2$ divisible by 3?

  Hint: In algebra we learn about factoring the difference of two squares, $x^2 - y^2 = (x-y)(x+y)$.

- Is $65,314,638,792$ divisible by 24?

  Hint: $24 = 2^3 * 3$

Divisibility by Powers of 5

Problems

- Is $1,234,567,890$ divisible by 5?

  Hint: Look at the last digit.

- How many 3 digit numbers are divisible by 5?

  Hint: The only numbers divisible by 5 are numbers which end in 5 or 0. So we want to know how many numbers between 99 and 1000 end in a 5 or a 0. Can you find an efficient way to count these?

- Find a divisibility test for 125. Use your test to decide if $1,234,567,890,000$ is visible by 750.

  Hint: Notice that 1000 is divisible by 125 and that $750 = 2 * 3 * 125$

- How do you test if a number is divisible by 5^n?

  Hint: Notice that 10^n is divisible by 5^n.

Divisibility by 7

Problems

- Is 623 divisible by 7?

  Hint: $62 - 2 * 3 = 56$.

- Is $1,234,567,890$ divisible by 7?

  Hint: At each step, remove the last digit, double it, and subtract it from what remains.

- Find a divisibility test for your favorite prime number.

  Hint: **See the website for a document about this.**

Divisibility by Powers of 10

Problems

- Is $1001^{10017} - 9812521809^2$ divisible by 10?

  Hint: Any number ending in 1, when raised to any power, still ends in a 1. Can you see why? Any number ending in 9, when squared, also ends in 1. Can you see why?

- How many zeros are there at the end of the decimal representation of 25! ? If this number is written in binary (base 2), how many zeros are at the end of it? Can you think of a base in which this number has only 1 zero at the end of it?

  Hint: To know how many zeros 25! has, we need to know how many powers of 10 it is divisible by. To know how many zeros are at the end of the binary representation of 25!, we need to know how many powers of 2 it is divisible by.

- If n is an integer, do n^5 and n always have the same last digit?

  Hint: They both have the same last digit if and only if $n^5 - n$ ends in 0, or in other words, $n^5 - n$ is divisible by 10.

- Is there an integer, n, so that $(n-1)! + 1$ is divisible by 10?

  Hint: The number $(n-1)!$ always ends in zero for any $n \geq 6$. Can you see why?

Divisibility by 11

Problems

- Is 1001 divisible by 11?

  Hint: $1 - 0 + 0 - 1 = 0$.

- Is $1,234,567,890$ divisible by 11?

  Hint: $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5$.

- Can the numbers $\{1, 2, 3, 4\}$ be arranged into a four digit number that is divisible by 11? What about the numbers $\{1, \ldots, 8\}$?

  Hint: Can you pair the numbers so that their differences are either 1 or -1 and then pair these ± 1 so that they cancel to give 0?

- It's easy to see that 1133 is divisible by 11. Using this, show very quickly that 3113 and $1,001,003,003,000$ are also divisible by 11.

  Hint: If a number is divisble by 11, which of its digits can be swapped to get a number still divisible by 11? How does adding 0's to a number effect its divisibility by 11?

- If a number has every one of its digits equal, under what conditions is that number divisible by 11?

  Hint: Consider the cases of even and odd numbers of digits separately.

More Problems and Extra Stuff

1. Prove that any product of k consecutive positive integers is divisible by k.

   Hint: Every other integer is divisible by 2. Every third integer is divisible by 3. And similarly, every kth integer is divisible by k. In other words, between two consecutive multiples of k there are exactly $k - 1$ integers which are not divisible by k.

2. If n is any integer, prove that $n^2 + n$ is always divisible by 2, that $n^3 - n$ is always divisible by 3, and that $n^5 - 5n^3 + 4n$ is always divisible by 5. For a given prime number, p, can you find a polynomial expression like these that is always divisible by p?

Hint: Factor the polynomials and use the previous problem.

3. Prove that $(p+1)^p - 1$ is divisible by p^2 if p is a prime number.

   Hint: Expand $(p+1)^p$ using the binomial theorem, and use the fact proved below that $\frac{p!}{k!(p-k)!}$ is an integer if $0 \le k \le p$. Is this integer also divisible by p?

4. Prove that $n^p - n$ is divisible by p if p is a prime number. This is known as Fermat's Little Theorem.

   Hint: Notice that $1^p - 1 = 0$ is divisible by p and use induction together with the binomial theorem.

The Binomial Theorem

Problems

- Check that $\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$.

  Hint: Combine the two fractions by finding a common denominator.

- Use the previous problem (and induction) to show that the coefficients in the binomial expansion $\left(\frac{n!}{k!(n-k)!}\right)$ are always integers.

  Hint: Define $C(n,k) = \frac{n!}{k!(n-k)!}$. Obviously $C(1,0) = C(1,1) = 1$ are both integers. As are $C(2,0) = C(2,2) = 1$ and $C(2,1) = 2$. Assume that $C(n,k)$ is an integer for all $1 \le n \le N-1$ and for $0 \le k \le n$ and use induction together with the result from the previous problem.