ARCHIMEDES AND THE ARBELOS¹

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The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

— G.H. Hardy, A Mathematician’s Apology

Problem 1. We will warm up on an easy problem: Show that traveling from $A$ to $B$ along the big semicircle is the same distance as traveling from $A$ to $B$ by way of $C$ along the two smaller semicircles.

Proof. The arc from $A$ to $C$ has length $\pi r/2$. The arc from $C$ to $B$ has length $\pi (1 - r)/2$. The arc from $A$ to $B$ has length $\pi/2$. $\square$

¹My notes are shamelessly stolen from notes by Tom Rike, of the Berkeley Math Circle available at http://mathcircle.berkeley.edu/BMC6/ps0506/ArbelosBMC.pdf.
If we draw the line tangent to the two smaller semicircles, it must be perpendicular to $AB$. (Why?) We will let $D$ be the point where this line intersects the largest of the semicircles; $X$ and $Y$ will indicate the points of intersection with the line segments $AD$ and $BD$ with the two smaller semicircles respectively (see Figure 2). Finally, let $P$ be the point where $XY$ and $CD$ intersect.

![Figure 2](image.png)

**Problem 2.** Now show that $XY$ and $CD$ are the same length, and that they bisect each other. What more can you say about these four points and their relationship with one another?

**Proof.** There is a famous theorem called Thales’ Theorem: if $A$, $B$, and $C$ are points on a circle, and $AB$ forms a diameter of the circle, then $\angle ABC$ is a right angle. This theorem implies that $\angle AXC$, $\angle ADB$, and $\angle CYB$ are all right angles. Therefore, since three of the angles in the quadrilateral $XDY C$ are right, then it is a rectangle. Hence its diagonals are equal, $CD = XY$; and bisect each other.

Also note that $X$, $D$, $Y$, and $C$ are all concircular (lie on the same circle). This is true for the vertices of any rectangle, but not true for any arbitrary four points in the plane. $\square$
Problem 3. Show that $XY$ is tangent to the small semicircles!

Proof. Let $G$ be the center of the arc $AXC$ and let $H$ be the center of the arc $CYB$. Then $GX = GC$, and $XP = PC$ (by the previous problem). Therefore, $\angle GX P = \angle GCP = \pi/2$. Thus, $XY$ is tangent at $X$. The case at $Y$ is similar.
Let $MN$ be a perpendicular bisector of $AB$ and a radius of the largest semicircle. Likewise, $EG$ and $FH$ are radii of the smaller semicircles, also perpendicular to $AB$. See Figure 3.

Problem 4. Show that $A$, $E$ and $N$ are colinear; as are $B$, $F$, and $N$. This is Proposition 1 in The Book of Lemmas.

Proof. $AG = GE$ so $\angle GAE = \pi/4$. $AM = MN$ so $\angle MAN = \pi/4$. Thus, $AEN$ is a line segment.

The case for $BFN$ is similar. \qed
Problem 5. Show that the area enclosed by the three semicircles is the same as the area of the circle with diameter $CD$ (see Figure 4). This is Proposition 4 in The Book of Lemmas.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Figure 4}
\end{figure}

Proof. Let $M$ be the center of the large semicircle. Let $d = CD$. It is easily seen through the Pythagorean Theorem applied to $\triangle MCD$ that $d^2 = r(1 - r)$.

So the area of the circle is

$$\text{area}(\bigcirc) = \frac{\pi d^2}{4} = \frac{\pi r (1 - r)}{4}.$$ 

The area of the Arbelos is easily checked by subtracting the areas of the small semicircles from the area of the large semicircle. We find the area is

$$\text{area}(\bigstar) = \frac{\pi}{8} - \frac{\pi r^2}{8} - \frac{\pi (1 - r)^2}{8} = \frac{\pi r (1 - r)}{4}.$$ 

The two areas agree. \qed
Now we inscribe two circles on either side of the line segment $CD$, as in Figure 5. These circles are called the Archimedean Twins.

![Figure 5](image)

**Problem 6.** Show that the Archimedean Twins have equal diameters. Find this diameter in terms of $r$. (This is Proposition 5 in *The Book of Lemmas*.)

**Problem 7.** Construct (with proof), the Archimedean Twins in a given Arbelos using a straightedge and compass (*i.e.*, Euclidean construction).

![Figure 6](image)

**Proof.** Construct $GE$ and $HF$ as perpendicular bisectors to $AC$ and $CB$, respectively. Construct the line $EH$, and let $Q$ be the intersection of $EH$ and $CD$. Then it is easily shown via similar triangles that $CQ = r(1 - r)/2$, which is the radius of the Twins. Construct the arc $RQS$, then the lines perpendicular to $AB$ at $R$ and $S$. Then construct the arcs centered at $A$ and $B$ with respective radii $AS$ and $BR$. The intersections of the perpendiculars through $R$ and $S$ with these arcs are the centers of the Twins. \(\square\)
Problem 8. Find the diameter of the circle tangent to all three semicircles that form the Arbelos (see Figure 7) in terms of $r$. This is Proposition 6 of *The Book of Lemmas.*
We can expand on this idea of inscribing circles in the Arbelos. Let $C_1$ be the circle given in the previous problem. Then let $C_n$ represent the $n$th circle in the chain as in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Figure 8}
\end{figure}

Problem 9 (Pappus). Show that the distance from the segment $AB$ to the center of the $n$th circle, $C_n$ in the chain is exactly $n \cdot d_n$, where $d_n$ is the diameter of $C_n$. (Hint: if you know how to use inversion in a circle orthogonal to $C_n$, this is much easier. However, Pappus originally solved this in a much different way.)

Problem 10. Show that the centers of the circles in the above chain all lie on an ellipse with foci at the centers of the two semicircles in which the chain is inscribed. In fact, any circle so inscribed between two such semicircles will have its center on this ellipse.
Now let’s turn our attention back to the Archimedean Twins. We will see that they occur a lot within and around the Arbelos (in fact, there are infinitely many such occurrences of the Twins!).

Our first recurrence of the Twins is simple enough: Inside each of the two smaller semicircles of the Arbelos, construct a similar Arbelos to the original (see Figure 9).

![Figure 9](image_url)

**Problem 11.** Show that the two middle semicircles of these new Arbelosi are actually Archimedean Twins.
Let $S$ and $T$ be the points where the first circle in our Pappus chain is tangent to each of the smaller semicircles in the Arbelos (see Figure 10).

![Figure 10](image)

**Problem 12.** Show that the circle formed by $S$, $T$, and $C$ is an Archimedean Twin.

**Problem 13.** Moreover, show that this circle passes through the point $P$, where $P$ is the intersection of $CD$ with the line connecting the zeniths of the small semicircles (see Figure 10).
Consider the circles with centers $A$ and $B$, and whose respective radii are $AC$ and $BC$. Construct a circle tangent to each of these, and inscribed in the large semicircle of the Arbelos (see Figure 11).

![Figure 11](image)

**Problem 14.** Show that this circle is an Archimedean Twin.
Let’s generalize the previous result. First, draw the line through the center of this most recent occurrence of a Twin, perpendicular to the baseline $AB$. This line is known as the Schoch line, and will allow us to find an infinite family of Archimedean Twins, known as the Woo circles.

![Figure 12](image1)

**Figure 12**

Let $m$ be any positive number. Construct two circles so their centers lie on the baseline of the Arbelos, and so their respective radii are $mr$ and $m(1 - r)$. Next consider the circle which is tangent to each of these, and whose center lies on the Schoch line. See Figure 13

![Figure 13](image2)

**Figure 13**

**Problem 15.** Show that this circle is also an Archimedean Twin (thus, we have an infinite family of Twins—one for each value of $m$).