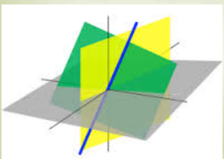
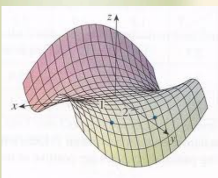


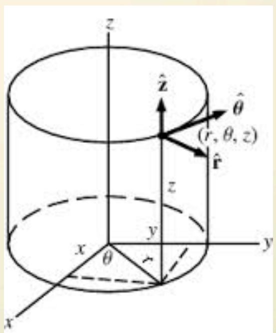
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

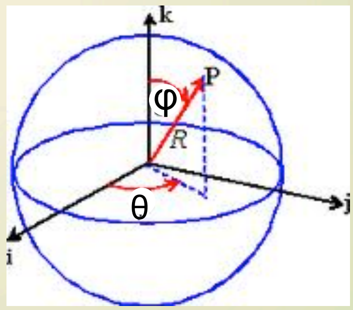
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



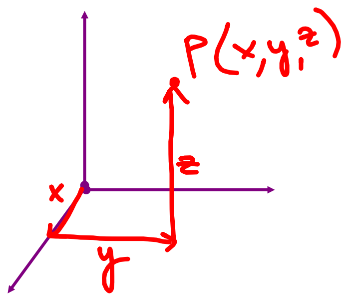
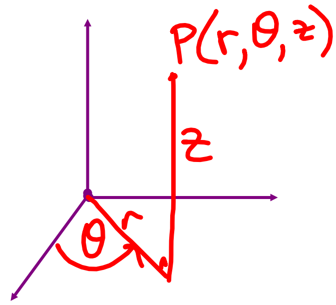
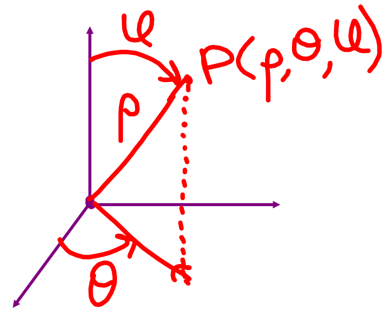
$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Cylindrical and Spherical Coordinates





We can describe a point, P, in three different ways.

Cartesian	Cylindrical	Spherical
		

Cylindrical Coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \text{from cyl. to Cartesian}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= y/x \\ z &= z \end{aligned}\right\} \text{from Cartesian to cylindrical}$$

$r \geq 0$, $\theta \in [0, 2\pi)$

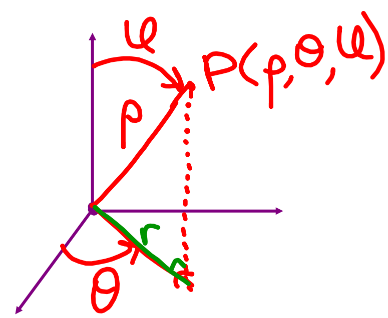
note: $\tan\theta = \frac{y}{x} \not\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Spherical Coordinates

$x = \underline{\rho\sin\varphi}\cos\theta$ $\quad$ $\rho = \sqrt{x^2+y^2+z^2}$

$y = \underline{\rho\sin\varphi}\sin\theta$ $\quad$ $\tan\theta = y/x$

$z = \rho\cos\varphi$ $\quad$ $\cos\varphi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

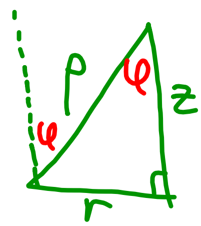


$\rho \geq 0$, $\theta \in [0, 2\pi)$, and $\varphi \in [0, \pi]$

pull out right Δ:

$r = \rho\sin\varphi$

$z = \rho\cos\varphi$

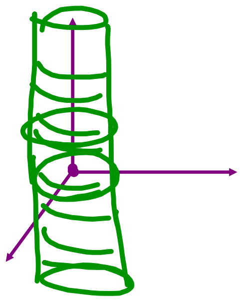


$\sin\varphi = \frac{r}{\rho}$

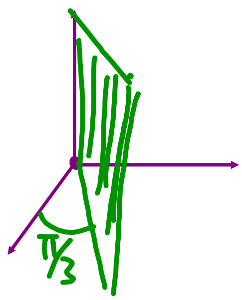
$\cos\varphi = \frac{z}{\rho}$

Easy Surfaces in Cylindrical Coordinates

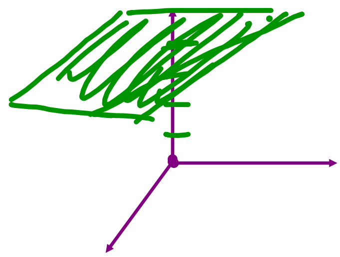
a) $r = 1$



b) $\theta = \pi/3$

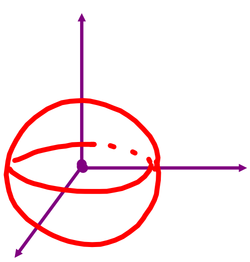


c) $z = 4$

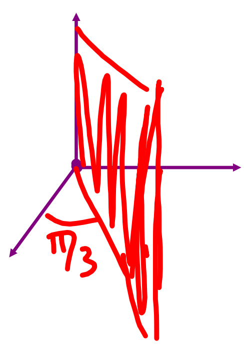


Easy Surfaces in Spherical Coordinates

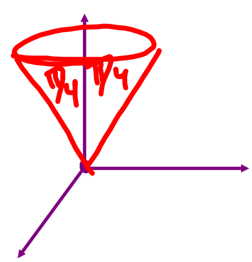
a) $\rho = 1$



b) $\theta = \pi/3$



c) $\varphi = \pi/4$



EX 1 Convert the coordinates as indicated

a) $\overset{r}{(3}, \overset{\theta}{\pi/3}, \overset{z}{-4)}$ from cylindrical to Cartesian.

$$x = r\cos\theta = 3\cos(\pi/3) = 3\left(\tfrac{1}{2}\right) = \tfrac{3}{2}$$

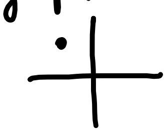
$$y = r\sin\theta = 3\sin(\pi/3) = 3\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{3\sqrt{3}}{2}$$

$$z = z = -4$$

$$\boxed{\left(3/2,\ 3\sqrt{3}/2,\ -4\right)}$$

b) $\overset{x}{(-2}, \overset{y}{2}, \overset{z}{3)}$ from Cartesian to cylindrical.

in xy-plane



$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{y}{x} \iff \tan\theta = \frac{2}{-2} = -1$$

$$\theta = 3\pi/4$$

$$z = z = 3$$

$$\boxed{\left(2\sqrt{2}, \tfrac{3\pi}{4}, 3\right)}$$

EX 2 Convert the coordinates as indicated

a) $\overset{\rho}{(8,}\ \overset{\theta}{\pi/4,}\ \overset{\varphi}{\pi/6)}$ from spherical to Cartesian.

$x = \rho\cos\theta\sin\varphi = 8\cos(\pi/4)\sin(\pi/6) = 8\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= 2\sqrt{2}$

$y = \rho\sin\theta\sin\varphi = 8\sin(\pi/4)\sin(\pi/6)$

$= 2\sqrt{2}$

$z = \rho\cos\varphi$

$= 8\cos(\pi/6) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$

$\boxed{(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})}$

b) $(\overset{x}{2\sqrt{3}},\ \overset{y}{6},\ \overset{z}{-4})$ from Cartesian to spherical.

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(2\sqrt{3})^2+6^2+4^2} = \sqrt{12+36+16}$

$= \sqrt{64} = 8$

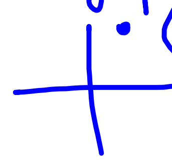
$\tan\theta = \frac{y}{x}$

$\cos\varphi = \frac{z}{\rho}$

$\tan\theta = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\tan\theta = \frac{\sqrt{3}/2}{1/2}$

in xy-plane:



(in Q1)

$\theta = \pi/3$

$\cos\varphi = \frac{-4}{8} = \frac{-1}{2}$

$\varphi \in [0, \pi]$ $\quad \varphi = 2\pi/3$

$\boxed{(8, \pi/3, 2\pi/3)}$

EX 3 Convert from cylindrical to spherical coordinates.

$\overset{r}{(1,}\ \overset{\theta}{\pi/2,}\ \overset{z}{1)}$

$\theta = \pi/2$

$\theta = \theta$

$r = \rho \sin\varphi$

$z = \rho\cos\varphi$

$\Rightarrow$ ① $\frac{r}{z} = \tan\varphi$

② $\rho = \frac{z}{\cos\varphi}$

① $\frac{1}{1} = \tan\varphi \qquad (\varphi \in [0, \pi])$

$\varphi = \pi/4$

② $\rho = \frac{1}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$

$\boxed{(\sqrt{2}, \pi/2, \pi/4)}$

EX 4 Make the required change in the given equation.

a) $x^2 - y^2 = 25$ to cylindrical coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$r^2\cos^2\theta - r^2\sin^2\theta = 25$

$r^2(\cos^2\theta - \sin^2\theta) = 25$

$r^2\cos(2\theta) = 25 \rightarrow r^2 = 25\sec(2\theta)$

b) $x^2 + y^2 - z^2 = 1$ to spherical coordinates.

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi - \rho^2\cos^2\varphi = 1$

$\rho^2\sin^2\varphi(\cos^2\theta + \sin^2\theta) - \rho^2\cos^2\varphi = 1$

$\rho^2(\sin^2\varphi - \cos^2\varphi) = 1$

$\rho^2(-\cos(2\varphi)) = 1$

$\Rightarrow \boxed{\rho^2 = -\sec(2\varphi)}$

c) $\rho = 2\cos\varphi$ to cylindrical coordinates.

$r = \rho\sin\varphi$

$z = \rho\cos\varphi$

$\rho^2 = 2\rho\cos\varphi$

$\rho^2 = 2z$

$r^2 + z^2 = \rho^2 \Longrightarrow r^2 + z^2 = 2z$

$r^2 = 2z - z^2$

EX 4 Make the required change in the given equation (continued).

d) $x + y + z = 1$ to spherical coordinates.

$$\rho\sin\varphi\cos\theta + \rho\sin\varphi\sin\theta + \rho\cos\varphi = 1$$

e) $r = 2\sin\theta$ to Cartesian coordinates.

$$r^2 = 2r\sin\theta$$

$$x^2 + y^2 = 2y$$

(cylinder)

$$x^2 + y^2 - 2y + 1 = 0 + 1$$

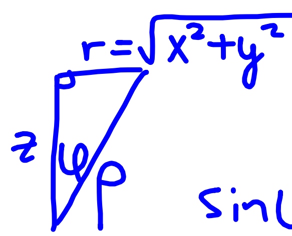
$$x^2 + (y-1)^2 = 1$$

f) $\rho\sin\theta = 1$ to Cartesian coordinates.

$$\rho\sin\varphi\sin\theta = \sin\varphi$$

$$y = \sin\varphi$$

$$y = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$



$$\sin\varphi = \frac{\sqrt{x^2+y^2}}{\rho}$$

$$\sin\varphi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$