Surfaces in Three-Space

\( f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \)

\( f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \)

\( \int_0^{2\pi} \int_0^2 \rho \rho^2 d\rho d\theta = \int_0^{2\pi} \left( \int_0^2 \rho^3 \right) d\theta \)

\( = \left( \frac{1}{4} \right)^4 \left( \frac{1}{2} \right)^1 = \frac{1}{32} \)
Quick Review of the Conic Sections

a) Parabola \[ y = x^2 \quad x = y^2 \]

b) Ellipse \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
\[ \text{if } y = 0, x = \pm a \]
\[ \text{if } x = 0, y = \pm b \]

a > b
a < b
a = b

c) Hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \]
Surfaces in Three-Space

The graph of a 3-variable equation which can be written in the form $F(x,y,z) = 0$ or sometimes $z = f(x,y)$ (if you can solve for $z$) is a surface in 3D. One technique for graphing them is to graph cross-sections (intersections of the surface with well-chosen planes) and/or traces (intersections of the surface with the coordinate planes).

We already know of two surfaces:

a) plane $Ax + By + Cz = D$

b) sphere $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

\[
F(x,y,z) = 0 \text{ (implicit)}
\]

\[
z = f(x,y) \text{ (explicit)}
\]
EX 1 Sketch a graph of \( z = x^2 + y^2 \) and \( x = y^2 + z^2 \).

If \( z = 0 \), \( x = 0, y = 0 \)

If \( z = 1 \), \( x^2 + y^2 = 1 \)

If \( z = 4 \), \( x^2 + y^2 = 4 \)

Cross-sections:
- In \( xy \)-plane, circles
- In \( xz \)-plane, parabolas
- In \( yz \)-plane, parabolas

(Shape will be same as in left drawing)
A cylinder is the set of all points on lines parallel to \( \ell \) that intersect C where C is a plane curve and \( \ell \) is a line intersecting C, but not in the plane of C.
A **Quadric Surface** is a 3D surface whose equation is of the second degree. The general equation is

\[ Ax^2 + By^2 + Cz^2 + Dxy + Ezx + Fyz + Gx + Hy + Iz + J = 0 , \]

given that \( A^2 + B^2 + C^2 \neq 0 \).

With rotation and translation, these possibilities can be reduced to two distinct types.

1) \( Ax^2 + By^2 + Cz^2 + J = 0 \) 
   - **all three quadratic terms; no linear terms**

2) \( Ax^2 + By^2 + Iz = 0 \) 
   - **only 2 quadratic terms; one linear term**

**Note:** a plane is not a quadric surface.
Basic Quadric Surfaces

ELLIPSOID

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Note: a sphere is an ellipsoid (w/ \( a = b = c \))

HYPERBOLOID OF ONE SHEET

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

axis

HYPERBOLOID OF TWO SHEETS

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

axis
**Type (1)**

**Elliptic Paraboloid**

\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

**Type (2)**

**Hyperbolic Paraboloid**

\[ z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \]

(looks like saddle or Pringles potato chip)

**Type (3)**

**Elliptic Cone**

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \]
EX 2 Name and sketch these graphs

a) $9x^2 + y^2 - z^2 = -4$
(no linear terms)

\[-\frac{9x^2}{4} - \frac{y^2}{4} + \frac{z^2}{4} = 1\]
hyperboloid of 2 sheets

b) $9x^2 + y^2 - z^2 = 4$
(no linear terms)

\[\frac{9x^2}{4} + \frac{y^2}{4} - \frac{z^2}{4} = 1\]
hyperboloid of one sheet about z-axis

c) $x^2 + 4y^2 - z = 0$

(? quadratic terms; one linear term)
about z-axis

\[z = x^2 + 4y^2\]
elliptic paraboloid
note: these are all cylinders

d) $x^2 + y^2 = 1$
(about $z$-axis)
(all cross-sections for planes $\parallel$ to $xy$-plane are unit circles)

e) $x^2 - y^2 = 25$
(about $z$-axis)
in 2-d $x^2 - y^2 = 25$ is a hyperbola

f) $z = y^2$
(about $x$-axis)
in 2-d