Vector-Valued Functions and Curvilinear Motion

\[ f_x = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \]

\[ f_y = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} \]

\[
\int_0^2 \int_0^{1/2} xy^3 \, dx \, dy = \int_0^2 \left( \int_0^{1/2} xy^3 \, dx \right) \, dy \\
= \int_0^2 \left[ \frac{1}{2} y^3 \right]_0^{1/2} \, dy \\
= \left[ \frac{1}{2} y^4 \right]_0^{1/2} = \frac{1}{2}
\]
A vector-valued function associates a vector output, \( \vec{F}(t) \),
to a scalar input.

\[
\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} = \langle f(t), g(t) \rangle \quad \text{(in 2D)}
\]

where \( f \) and \( g \) are real-valued functions of \( t \)

or

\[
\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \quad \text{(in 3D)}.
\]

**Definition** \( \lim_{t \to c} \vec{F}(t) = \vec{L} \) means that for every \( \varepsilon > 0 \) there is a corresponding \( \delta > 0 \) such that \( \| \vec{F}(t) - \vec{L} \| < \varepsilon \), provided

\[
0 < |t - c| < \delta \quad \text{, i.e.}
\]

\[
0 < |t - c| < \delta \Rightarrow \| \vec{F}(t) - \vec{L} \| < \varepsilon .
\]
Theorem A  Let \( \vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} \). Then \( \vec{F} \) has a limit at \( c \) iff \( f \) and \( g \) have limits at \( c \) and

\[
\lim_{t \to c} \vec{F}(t) = \left[ \lim_{t \to c} f(t) \right] \hat{i} + \left[ \lim_{t \to c} g(t) \right] \hat{j}
\]

Continuity \( \Rightarrow \) \( \vec{F}(t) \) is continuous at \( t = c \) if \( \lim_{t \to c} \vec{F}(t) = \vec{F}(c) \).

Derivative \( \Rightarrow \) \( \vec{F}'(t) = \lim_{h \to 0} \frac{\vec{F}(t + h) - \vec{F}(t)}{h} \)

Note: basically expand these notions from \( f(t) \) of one variable to vector-valued \( f(t) \) by doing everything component by component.
Differentiation Formulas

$\tilde{F}(t)$ & $\tilde{G}(t)$ are differentiable
$c \in \mathbb{R}$

$h(t)$ is differentiable

1) $D_t[\tilde{F}(t) + \tilde{G}(t)] = \tilde{F}'(t) + \tilde{G}'(t)$
2) $D_t[c\tilde{F}(t)] = c\tilde{F}'(t)$
3) $D_t[h(t) \cdot \tilde{F}(t)] = h(t) \cdot \tilde{F}'(t) + \tilde{F}(t) \cdot h'(t)$
4) $D_t[\tilde{F}(t) \cdot \tilde{G}(t)] = \tilde{F}(t) \cdot \tilde{G}'(t) + \tilde{F}'(t) \cdot \tilde{G}(t)$
5) $D_t[\tilde{F}(h(t))] = \tilde{F}'(h(t)) \cdot h'(t)$

Integration Formula

$$\int \tilde{F}(t) dt = \left[ \int f(t) dt \right] \hat{i} + \left[ \int g(t) dt \right] \hat{j} \quad \text{(in Zd)}$$
Ex 1 Find \( \lim_{t \to \infty} \left[ \frac{t \sin t}{t^2} \hat{i} - \frac{7t^3}{t^3 - 3t} \hat{j} \right] \).

\[
= \left( \lim_{t \to \infty} \left( \frac{t \sin t}{t^2} \right) \right) \hat{i} - \left( \lim_{t \to \infty} \frac{7t^3}{t^3 - 3t} \right) \hat{j}
\]

\[
= \left( \lim_{t \to \infty} \frac{\sin t}{t} \right) \hat{i} - \left( \lim_{t \to \infty} \frac{7t^3}{t^3 - 3t} \right) \hat{j}
\]

\[
= 0 \hat{i} - 7 \hat{j} = \langle 0, -7 \rangle
\]

EX 2 Find \( \vec{F}'(x) \) and \( \vec{F}''(x) \) for \( \vec{F}(x) = (e^x + e^{-x^2}) \hat{i} + \cos(2x) \hat{j} \).

\[
\vec{F}'(x) = (e^x + e^{-x^2} (-2x)) \hat{i} + (-2 \sin(2x)) \hat{j}
\]

\[
\vec{F}''(x) = (e^x - (2e^{-x^2} + 2x e^{-x^2} (-2x))) \hat{i}
+ (-4 \cos(2x)) \hat{j}
= (e^x - 2e^{-x^2} + 4x^2 e^{-x^2}) \hat{i} - 4 \cos(2x) \hat{j}
\]
EX 3  \( \vec{f}(y) = (\tan^2 y)\hat{i} + \sin^2(\tan^2 y)\hat{j} + 3y\hat{k} \)
Finding \( \vec{f}'(y) \).

\[
\vec{f}'(y) = (2\tan y \sec^2 y)\hat{i}
+ (2 \sin(\tan^2 y) \cos(\tan^2 y) 
\cdot 2 \tan y \sec^2 y)\hat{j} + 3\hat{k}
\]
\( \mathbf{r}(t) \) is position vector at any time \( t \) along a curve given by
\[
x = x(t) \quad \text{and} \quad y = y(t).
\]

\[
\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}
\]

\[
\mathbf{v}(t) = \mathbf{r}'(t) = x'(t) \mathbf{i} + y'(t) \mathbf{j}
\] velocity

\[
\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = x''(t) \mathbf{i} + y''(t) \mathbf{j}
\] acceleration

Note: velocity is a vector-valued function, speed is a number that is magnitude of velocity at some time \( t \).
EX 4 Given \( \mathbf{r}(t) = (4 \sin t) \hat{i} + (8 \cos t) \hat{j} \),

a) Find \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \).

\[
\mathbf{v}(t) = \mathbf{r}'(t) = (4 \cos t) \hat{i} + (-8 \sin t) \hat{j} \\
\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -4 \sin t \hat{i} - 8 \cos t \hat{j} = -\mathbf{r}(t)
\]

b) Find the speed when \( t = \pi/4 \).

\[
\mathbf{v}(\pi/4) = (4 \cos (\pi/4)) \hat{i} - (8 \sin (\pi/4)) \hat{j} = 2\sqrt{2} \hat{i} - 4\sqrt{2} \hat{j}
\]

\[
\text{Speed} = ||\mathbf{v}(\pi/4)|| = \sqrt{(2\sqrt{2})^2 + (-4\sqrt{2})^2} = \sqrt{8 + 32} = \sqrt{40} = 2\sqrt{10}
\]

c) Sketch a portion of the graph of \( \mathbf{r}(t) \) containing the position \( P \) of the particle at \( t = \pi/4 \). Draw \( \mathbf{v} \) and \( \mathbf{a} \) at \( P \) as well.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \mathbf{r} = 4 \sin t \hat{i} + 8 \cos t \hat{j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( &lt;0,8&gt; )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>( &lt;4,0&gt; )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( &lt;2\sqrt{2},4\sqrt{2}&gt; )</td>
</tr>
</tbody>
</table>

\( \mathbf{v} = <2\sqrt{2},-4\sqrt{2}> \)
EX 5 Suppose that an object moves around a circle with center at \((0,0)\) and radius \(r\) at a constant angular speed of \(\omega\) radians/sec.

If its initial position is \((0,r)\), find its acceleration.

\[
\vec{P}(t) = r \sin(\omega t) \hat{i} + r \cos(\omega t) \hat{j}
\]

**Position vector**

**Check my guess:**

At \(t=0\),

\[
\vec{P}(0) = r \sin 0 \hat{i} + r \cos 0 \hat{j} = \langle 0, r \rangle
\]

**This is plugging at \(P_0\).**

Find \(\vec{a}(t)\).

\[
\vec{v}(t) = \vec{v}(0) = \omega r \cos(\omega t) \hat{i} - \omega r \sin(\omega t) \hat{j}
\]

\[
\vec{a}(t) = \vec{v}'(t) = -\omega^2 r \sin(\omega t) \hat{i} - \omega^2 r \cos(\omega t) \hat{j}
\]

\[
\vec{a}(t) = -\omega^2 \vec{v}(t)
\]