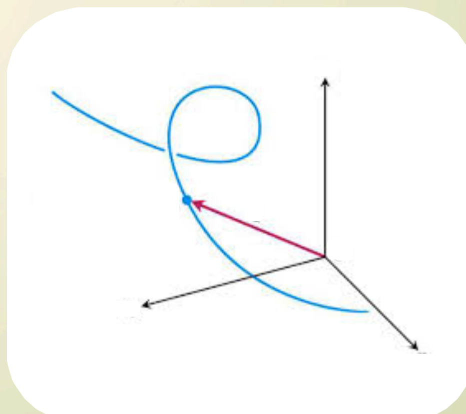
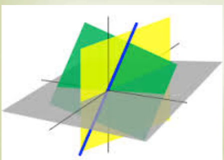
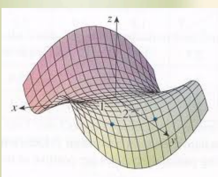
Vector-Valued Functions
and
Curvilinear Motion





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

A vector-valued function associates a vector output, $\vec{F}(t)$,

to a scalar input.

i.e $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} = \langle f(t), g(t) \rangle$ (in 2D)

where f and g are real-valued functions of t

or $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ (in 3D).

Definition $\lim_{t \to c} \vec{F}(t) = \vec{L}$ means that for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that $\left\| \vec{F}(t) - \vec{L} \right\| < \varepsilon$, provided $0 < |t - c| < \delta$, i.e.

$$0 < |t - c| < \delta \Rightarrow \left\| \vec{F}(t) - \vec{L} \right\| < \varepsilon .$$

Theorem A Let $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j}$. Then $\vec{F}$ has a limit at c iff f and g have limits at c *and*

$$\lim_{t \to c} \vec{F}(t) = \left[\lim_{t \to c} f(t)\right]\hat{i} + \left[\lim_{t \to c} g(t)\right]\hat{j}$$

Continuity $\Rightarrow$ $\vec{F}(t)$ is continuous at $t = c$ if $\lim_{t \to c} \vec{F}(t) = \vec{F}(c)$.

Derivative $\Rightarrow$ $\vec{F}\,'(t) = \lim_{h \to 0} \dfrac{\vec{F}(t+h) - \vec{F}(t)}{h}$

Note: basically expand these notions from fns of one variable to vector-valued fns by doing everything component by component.

$\vec{F}(t)$ & $\vec{G}(t)$ are differentiable

$c \in \Re$

$h(t)$ is differentiable

Differentiation Formulas

derivative is still a linear operator

1) $D_t[\vec{F}(t)+\vec{G}(t)] = \vec{F}'(t)+\vec{G}'(t)$

2) $D_t[c\vec{F}(t)] = c\vec{F}'(t)$

product rules

3) $D_t[h(t)\cdot\vec{F}(t)] = h(t)\cdot\vec{F}'(t)+h'(t)\cdot\vec{F}(t)$

4) $D_t[\vec{F}(t)\cdot\vec{G}(t)] = \vec{F}(t)\cdot\vec{G}'(t)+\vec{F}'(t)\cdot\vec{G}(t)$

chain rule

5) $D_t[\vec{F}(h(t))] = \vec{F}'(h(t))\cdot h'(t)$

Integration Formula

$$\int \vec{F}(t)dt = \left[\int f(t)dt\right]\hat{i} + \left[\int g(t)dt\right]\hat{j}$$ (in 2d)

Ex 1 Find $\lim_{t\to\infty}\left[\frac{t\sin t}{t^2}\hat{i}-\frac{7t^3}{t^3-3t}\hat{j}\right]$.

$$=\left(\lim_{t\to\infty}\left(\frac{\cancel{t}\sin t}{t^{\cancel{2}}}\right)\right)\hat{i}-\left(\lim_{t\to\infty}\frac{7t^3}{t^3-3t}\right)\hat{j}$$

$$=\left(\lim_{t\to\infty}\frac{\sin t}{t}\right)\hat{i}-\left(\lim_{t\to\infty}\frac{7\cancel{t^3}}{\cancel{t^3}}\right)\hat{j}$$

$$=0\hat{i}-7\hat{j}=\langle 0,-7\rangle$$

EX 2 Find $\vec{F}'(x)$ and $\vec{F}''(x)$ for $\vec{F}(x)=(e^x+e^{-x^2})\hat{i}+\cos(2x)\hat{j}$.

$$\vec{F}'(x)=\left(e^x+e^{-x^2}(-2x)\right)\hat{i}+\left(-2\sin(2x)\right)\hat{j}$$

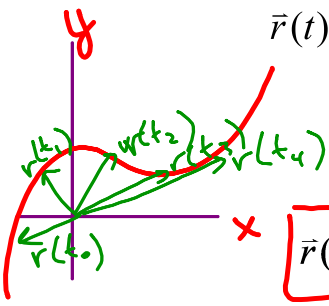
$$\vec{F}''(x)=\left(e^x-\left(2e^{-x^2}+2x\,e^{-x^2}(-2x)\right)\right)\hat{i}$$
$$+\left(-4\cos(2x)\right)\hat{j}$$
$$=\left(e^x-2e^{-x^2}+4x^2e^{-x^2}\right)\hat{i}-4\cos(2x)\hat{j}$$

EX 3 $\vec{f}(y) = (\tan^2 y)\hat{i} + \sin^2(\tan^2 y)\hat{j} + 3y\,\hat{k}$

Find $\vec{f}\,'(y)$.

(3-d)

$$\vec{f}\,'(y) = (2\tan y \sec^2 y)\hat{i} + \left(2\sin(\tan^2 y)\cos(\tan^2 y) \cdot 2\tan y \sec^2 y\right)\hat{j} + 3\hat{k}$$

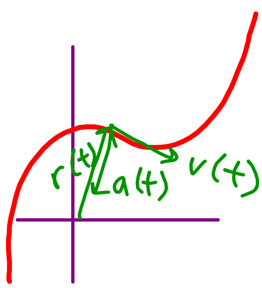


$\vec{r}(t)$ is position vector at any time t along a curve given by

$x = x(t)$ and $y = y(t)$.

$$\boxed{\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}}$$

$$\vec{v}(t) = \vec{r}\,'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$ velocity



$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = x''(t)\hat{i} + y''(t)\hat{j}$$ acceleration

note: velocity is a vector-valued fn; speed is a number that is magnitude of velocity at some time t.

EX 4 Given $\vec{r}(t) = (4\sin t)\hat{i} + (8\cos t)\hat{j}$,

a) Find $\vec{v}(t)$ and $\vec{a}(t)$.

$$\vec{v}(t) = \vec{r}\,'(t) = (4\cos t)\hat{i} + (-8\sin t)\hat{j}$$

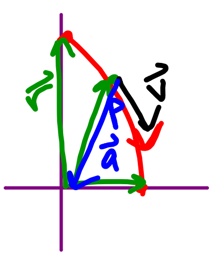
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = -4\sin t\,\hat{i} - 8\cos t\,\hat{j}$$

$$= -\vec{r}(t)$$

b) Find the speed when $t = \pi/4$.

$$\vec{v}(\pi/4) = (4\cos(\pi/4))\hat{i} - (8\sin(\pi/4))\hat{j}$$

$$= 2\sqrt{2}\,\hat{i} - 4\sqrt{2}\,\hat{j}$$



$$\text{speed} = \|\vec{v}(\pi/4)\| = \sqrt{(2\sqrt{2})^2 + (-4\sqrt{2})^2}$$

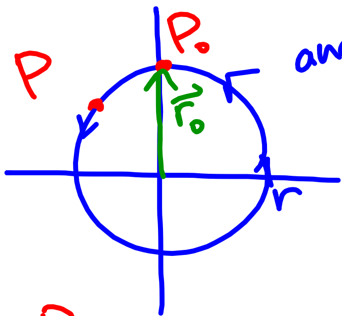
$$= \sqrt{8+32} = \sqrt{40} = 2\sqrt{10}$$

c) Sketch a portion of the graph of $\vec{r}(t)$ containing the position P of the particle at $t = \pi/4$. Draw $\vec{v}$ and $\vec{a}$ at P as well.

t	$\vec{r} = 4\sin t\,\hat{i} + 8\cos t\,\hat{j}$
0	$\langle 0, 8 \rangle$
$\pi/2$	$\langle 4, 0 \rangle$
$\pi/4$	$\langle 2\sqrt{2}, 4\sqrt{2} \rangle$

$$\vec{v} = \langle 2\sqrt{2}, -4\sqrt{2} \rangle$$

EX 5 Suppose that an object moves around a circle with center at *(0,0)* and radius r at a constant angular speed of ω radians/sec. If its initial position is *(0,r),* find its acceleration.



angular speed of ω

P can be represented by $(r\cos\theta, r\sin\theta)$

in our case $\theta = \omega t$

position vector

$$\vec{r}(t) = r\sin(\omega t)\hat{\imath} + r\cos(\omega t)\hat{\jmath}$$

check my guess:

at $t=0$,

$$\vec{r}(0) = r\sin 0\,\hat{\imath} + r\cos 0\,\hat{\jmath}$$

$$\vec{r}(0) = 0\hat{\imath} + r\hat{\jmath} = \langle 0, r\rangle$$

this is ptng at P_0. ✓

find $\vec{a}(t)$.

$$\vec{r}(t) = r\sin(\omega t)\hat{\imath} + r\cos(\omega t)\hat{\jmath}$$

$$\vec{r}\,'(t) = \vec{v}(t) = r\omega\cos(\omega t)\hat{\imath} - r\omega\sin(\omega t)\hat{\jmath}$$

$$\vec{a}(t) = \vec{v}\,'(t) = -r\omega^2\sin(\omega t)\hat{\imath} - r\omega^2\cos(\omega t)\hat{\jmath}$$

$$= -\omega^2\left(r\sin(\omega t)\hat{\imath} + r\cos(\omega t)\hat{\jmath}\right)$$

$$\boxed{\vec{a}(t) = -\omega^2\,\vec{r}(t)}$$