Surface Integrals

Let $G$ be defined as some surface, $z = f(x,y)$.

The surface integral is defined as

$$\int \int_G g(x,y,z) \, dS,$$

where $dS$ is a "little bit of surface area."

To evaluate we need this Theorem:

Let $G$ be a surface given by $z = f(x,y)$ where $(x,y)$ is in $R$, a bounded, closed region in the $xy$-plane.

If $f$ has continuous first-order partial derivatives and $g(x,y,z) = g(x,y,f(x,y))$ is continuous on $R$, then

$$\int \int_G g(x,y,z) \, dS = \int \int_R g(x,y,f(x,y)) \sqrt{f_x^2 + f_y^2 + 1} \, dy \, dx.$$

EX 1 Evaluate $\int \int_G g(x,y,z) \, dS$ given by $g(x,y,z) = x$, and $G$ is the plane $x + y + 2z = 4$, $x \in [0,1]$, $y \in [0,1]$. 
EX 2 Evaluate \( \iint_G (2y^2+z) \, dS \) where \( G \) is the surface 
\[ z = x^2 - y^2, \text{ with } R \text{ given by } 0 \leq x^2 + y^2 \leq 1. \]

EX 3 Evaluate \( \iint_G g(x,y,z) \, dS \) where \( g(x,y,z) = z \) and \( G \) is the tetrahedron bounded by the coordinate planes and the plane \( 4x + 8y + 2z = 16. \)
Theorem

Let $G$ be a smooth, two-sided surface given by $z = f(x,y)$, where $(x,y)$ is in $\mathbb{R}$ and let $\hat{n}$ denote the upward unit normal on $G$. If $f$ has continuous first-order partial derivatives and $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is a continuous vector field, then the flux of $\mathbf{F}$ across $G$ is given by

$$\text{flux } \mathbf{F} = \iint_G \mathbf{F} \cdot \hat{n} \, dS = \iint_R [-Mf_x - Nf_y + P] \, dx \, dy.$$

EX 4 Evaluate the flux of $\mathbf{F}$ across $G$ where

$$\mathbf{F}(x,y,z) = (9-x^2)\hat{j}$$

and $G$ is the part of the plane $2x + 3y + 6z = 6$ in the first octant.