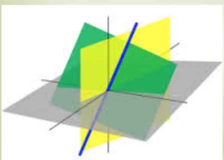
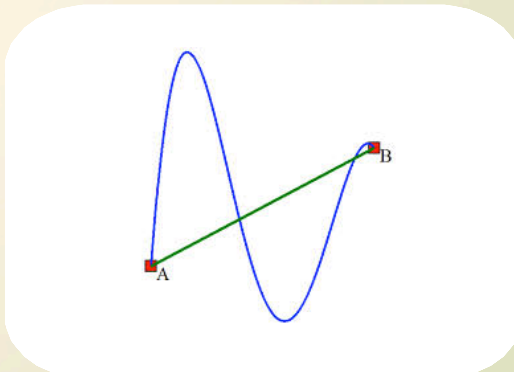
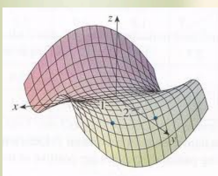
Independence of Path





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$

Recall the Fundamental Theorem of Calculus.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

We would like an analogous theorem for line integrals.

<u>Fundamental Theorem of Line Integrals</u>

Let C be the curve given by the parameterization $\vec{r}(t),\ t \in [a,b]$, such that $\vec{r}(t)$ is differentiable. If $f(\vec{r})$ is continuously differentiable on an open set containing C,

then

$$\boxed{\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

EX 1 Find work done by ∇f along a curve going from
$\underbrace{(1,1,1) \text{ to } (4,-1,2)}_{C}$, given $f(r) = \dfrac{c}{\|\vec{r}\|}$ $\boxed{\nabla f = \dfrac{-c\vec{r}}{\|\vec{r}\|}}$.

$C: \vec{r}(t) = ?$

$$\begin{cases} x = 1 + 3t \\ y = 1 + -2t \\ t = 1 + t \end{cases} \quad t \in [0,1]$$

$\vec{r}(t) = \langle 1+3t, 1-2t, 1+t \rangle$

$\vec{r}\,'(t) = \langle 3, -2, 1 \rangle$

$$W = \int_C \nabla f \cdot d\vec{r} = \int_0^1 \frac{-c\vec{r}}{\|\vec{r}\|} \cdot \langle 3,-2,1\rangle \, dt$$

$$= -c \int_0^1 \frac{\langle 1+3t, 1-2t, 1+t \rangle \cdot \langle 3,-2,1 \rangle}{\sqrt{(1+3t)^2 + (1-2t)^2 + (1+t)^2}} \, dt$$

$$= -c \int_0^1 \frac{(3+9t-2+4t+1+t)}{\sqrt{3+4t+14t^2}} \, dt$$

$$= -c \int_0^1 \frac{14t+2}{\sqrt{14t^2+4t+3}} \, dt$$

$u = 14t^2 + 4t + 3$

$du = (28t + 4)\, dt$

$\frac{1}{2} du = (14t+2)\, dt$

$t = 0, \ u = 3$

$t = 1, \ u = 21$

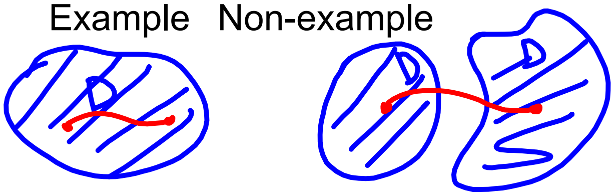
$$= -c\left(\frac{1}{2}\right) \int_3^{21} \frac{du}{\sqrt{u}}$$

$$= -\frac{c}{2}\left(2\sqrt{u}\right)\Big|_3^{21} = -c\left(\sqrt{21} - \sqrt{3}\right)$$

$$= \sqrt{3}\,c\left(1 - \sqrt{7}\right)$$

A set, D, is called a <u>Path-Connected Set</u> if any 2 points in D can be joined by a piece-wise smooth curve lying entirely in D.

Example Non-example



What does it mean to be independent of path?

$\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path in D if for any 2 pts, A and B in D, the line integral has the same value for all paths C in D, positively oriented from A to B.

<u>Independence of Path Theorem</u>

Let $\vec{F}(\vec{r})$ be continuous on an open connected set D.

Then $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of any path, C, in D iff $\Leftrightarrow$ $\vec{F}(\vec{r}) = \nabla f(\vec{r})$ for some $f(\vec{r})$ (scalar function),
i.e. if $\vec{F}(\vec{r})$ is a conservative vector field on D.

(f is called the potential fn)

Equivalent Conditions for Line Integrals

Let $\vec{F}(\vec{r})$ be continuous on an open connected set D.
The following statements are equivalent.

$\Longleftrightarrow$ a) $\vec{F} = \nabla f$ for some f (i.e. $\vec{F}$ is conservative on D).

$\Longleftrightarrow$ b) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of the path, C, in D.

$\Longleftrightarrow$ c) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ for every closed path in D.

(b) $\Rightarrow$ (c) because for any closed loop in D
start A = end pt B.

Theorem

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ with M, N, P continuously differentiable on a ball, D.

Then $\vec{F}$ is conservative $\Leftrightarrow$ $\nabla \times \vec{F} = \vec{0}$.

Note:

If $\vec{F} = M\hat{i} + N\hat{j}$

then $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

(how to compute 2-d "curl")

and $$\nabla \times \vec{F} = \vec{0} \Rightarrow \boxed{\frac{dN}{dx} = \frac{dM}{dy}}$$

Note:

$\vec{F}$ conservative $\Rightarrow \vec{F} = \nabla f$

$\Longleftrightarrow M\hat{i} + N\hat{j} + P\hat{k} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

for some f. Since derivatives are continuous for M, N, P, then

$$M_y = f_{xy}, \qquad N_x = f_{yx}$$

but $f_{xy} = f_{yx}$, then $M_y = N_x$

similarly $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \Leftrightarrow \nabla \times \vec{F} = \vec{0}$ //

EX 2 Is $\vec{F} = \underbrace{(12x^2 + 3y^2 + 5y)}_{M}\,\hat{i} + \underbrace{(6xy - 3y^2 + 5x)}_{N}\,\hat{j}$ conservative?

$$\frac{\partial M}{\partial y} = 6y+5$$

$$\frac{\partial N}{\partial x} = 6y+5$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow \vec{F}$ is conservative

EX 3 Using $\vec{F}$ from Example 1, find f such that $\vec{F} = \nabla f$.

know $\vec{F} = \nabla f \Leftrightarrow M = f_x,\ N = f_y$

$$f_x = 12x^2+3y^2+5y$$

$$f = \int (12x^2+3y^2+5y)\,dx$$

① $f = \frac{12x^3}{3} + 3y^2x + 5yx + C(y)$

know $f_y = 6xy - 3y^2 + 5x$ (since $f_y = N$)

find $f_y = 6yx + 5x + C'(y)$

equate: $6xy - 3y^2 + 5x = 6yx + 5x + C'(y)$

$$-3y^2 = C'(y)$$

$$C(y) = \int -3y^2\,dy = -y^3 + K$$

② $\Rightarrow \boxed{f = 4x^3 + 3xy^2 + 5xy - y^3 + K}$

EX 4 Using $\vec{F} = (12x^2 + 3y^2 + 5y)\hat{i} + (6xy - 3y^2 + 5x)\hat{j}$

calculate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ where C is any path from $(0,0)$ to $(2,1)$.

know from Ex 2 + 3 work:

$\vec{F}$ is conservative and

$f(x,y) = 4x^3 + 3y^2x + 5xy - y^3 + K$

we also know then that $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of path

and from Fundamental Thm of Line Integrals,

$\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$.

$$\Rightarrow \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(2,1) - f(0,0)$$

$$= \left(4(2^3) + 3(1^2 \cdot 2) + 5(2 \cdot 1) - 1^3\right) - 0$$

$$= 4(8) + 6 + 10 - 1 = 47$$

EX 5 Show that the line integral $\displaystyle\int_C \underbrace{((yz+1)dx + (xz+1)dy + (xy+1)dz)}_{\vec{F}\cdot d\vec{r}}$

is independent of path and evaluate the integral, where C is a curve from (0,1,0) to (1,1,1).

$\vec{F}(x,y,z) = (yz+1)\hat{\imath} + (xz+1)\hat{\jmath} + (xy+1)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+1 & xz+1 & xy+1 \end{vmatrix}$$

$= \hat{\imath}(x-x) + -\hat{\jmath}(y-y) + \hat{k}(z-z) = \vec{0}$

$\Rightarrow \vec{F}$ conservative vector field.

$\Rightarrow f$ exists, such that $\vec{F} = \nabla f$.

$f_x = yz+1 \Rightarrow$ ① $f = \int (yz+1)dx = yzx + x + C(y,z)$

also know $f_y = xz+1$

and from ①, we get $f_y = zx + \dfrac{\partial C(y,z)}{\partial y}$

equate: $xz+1 = xz + \dfrac{\partial C(y,z)}{\partial y}$

$\Leftrightarrow 1 = \dfrac{\partial C(y,z)}{\partial y}$

$C(y,z) = \int 1\, dy = y + D(z)$

$\Rightarrow$ ② $f = xyz + x + y + D(z)$

we know $f_z = xy+1$

and from ②, $f_z = xy + D'(z)$

equate: $xy+1 = xy + D'(z)$

$\Rightarrow 1 = D'(z) \Rightarrow D(z) = \int 1\, dz = z + K$

③ $\Rightarrow f(x,y,z) = xyz + x + y + z + K$

Now evaluate the line integral:

$\displaystyle\int_C \vec{F}(\vec{r})\cdot d\vec{r} = f(1,1,1) - f(0,1,0)$

$= (1+1+1+1) - (0+0+1+0)$

$= 3$

EX 6 Let $\vec{F} = (1 + 2xy\sin(x^2y))\hat{i} + (1 + x^2\sin(x^2y))\hat{j}$

① Is $\vec{F}$ conservative? (check $\nabla \times \vec{F}$ to see if it's $\vec{0}$)

② If yes, then find f such that $\vec{F} = \nabla f$.

① $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 1+2xy\sin(x^2y) & 1+x^2\sin(x^2y) & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(2x\sin(x^2y) + 2x^3y\cos(x^2y) - (2x\sin(x^2y) + 2x^3y\cos(x^2y))\right)$$

$$= \vec{0}$$

$\Rightarrow \vec{F}$ is conservative.

② find f such that $\vec{F} = \nabla f$.

we know $f_x = 1 + 2xy\sin(x^2y)$

$\Rightarrow f = \int (1 + 2xy\sin(x^2y))\,dx$

① $f = x - \cos(x^2y) + C(y)$

we also know $f_y = 1 + x^2\sin(x^2y)$

and from ①, $f_y = x^2\sin(x^2y) + C'(y)$

$\Rightarrow 1 + x^2\sin(x^2y) = x^2\sin(x^2y) + C'(y)$

$\Leftrightarrow 1 = C'(y) \Rightarrow C(y) = \int 1\,dy = y + K$

② $\Rightarrow \boxed{f(x,y) = x - \cos(x^2y) + y + K}$

EX 7 Evaluate $\int_{(0,0)}^{(1,\pi/2)} \left(\underbrace{e^x \sin y}_{M} dx + \underbrace{e^x \cos y}_{N} dy\right)$.

1) Prove $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ is conservative.

2) Find $f(x,y)$ such that $\vec{F} = \nabla f$.

3) Using f and Fund. Thm of Line Integrals, $\int_C \vec{F}\cdot d\vec{r} = f(1,\pi/2) - f(0,0)$.

1) $\frac{\partial M}{\partial y} = e^x \cos y$, $\frac{\partial N}{\partial x} = e^x \cos y \Rightarrow \nabla \times \vec{F} = \vec{0}$

i.e. $\vec{F}$ conservative.

2) $f_x = e^x \sin y \Rightarrow f = \int (e^x \sin y)\,dx$

① $f = e^x \sin y + C(y)$

we know $f_y = e^x \cos y$

and from ① $f_y = e^x \cos y + C'(y)$

equate: $e^x \cos y = e^x \cos y + C'(y)$

$0 = C'(y)$

$\Rightarrow C(y) = \int 0\,dy = k$

$\Longrightarrow$ ② $\boxed{f = e^x \sin y + k}$

3) $\int_{(0,0)}^{(1,\frac{\pi}{2})} \vec{F}\cdot d\vec{r} = f(1,\pi/2) - f(0,0)$

$= \left(e^1 \sin\left(\frac{\pi}{2}\right) + k\right) - \left(e^0 \sin 0 + k\right)$

$= e(1) - 0$

$= e$