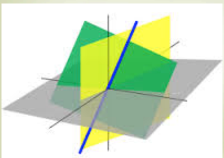
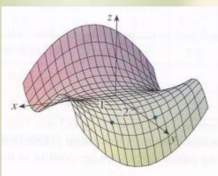


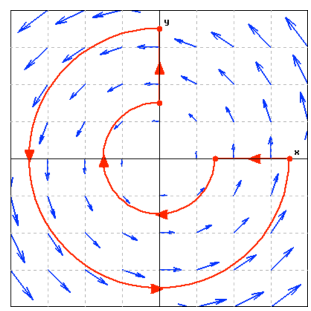
$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

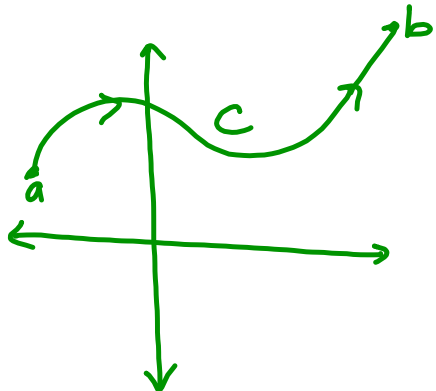


$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y} dy$$
$$= \int_0^1 \frac{(2y)^2}{2}\,y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Line Integrals

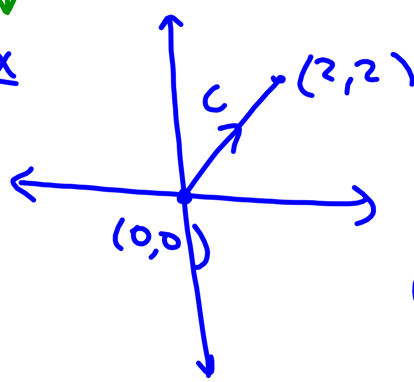


Let's review parameterization of curves.



C can be represented in infinitely many different parameterizations

ex



different parameterizatns

① $y=x$, $\begin{cases} x=t \\ y=t \end{cases}$, $t \in [0,2]$

② $\begin{cases} x=4t \\ y=4t \end{cases}$, $t \in [0, \frac{1}{2}]$

$\Longleftrightarrow \vec{r}(t) = \langle 4t, 4t \rangle = 4t\hat{i} + 4t\hat{j}$

$t \in [0, \frac{1}{2}]$

③ $\vec{r}(t) = \langle t^3+1, t^3+1 \rangle$, $t \in [-1, 1]$

The length of a parameterized curve in 2-D $(x(t),y(t)),\ t \in [a,b]$ is given by

$$\boxed{L = \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{ds}}$$

In 3-D if $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, (a parameterization of our curve, C)

then the length of a curve is

$$L = \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}_{ds} = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

$$\vec{r}\,'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$$

$$\rightarrow \|\vec{r}\,'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Suppose $f(x,y)$ is a function whose domain contains the curve

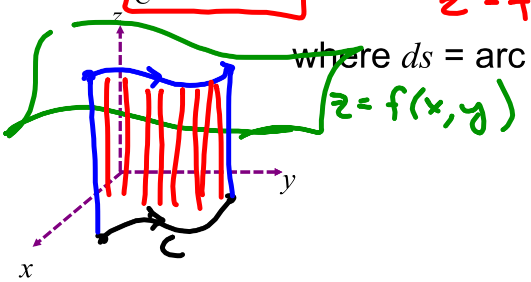
$$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}\,, \qquad t \in [a,b]\,.$$

(C has this parameterization of $\vec{r}(t)$)

The line integral of f along the curve C from a to b is defined

as $\boxed{\int_C f(x,y)\,ds}$ (area under the surface $z=f(x,y)$ and above curve C)

where ds = arc length differential.



We know that $\quad ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$ (in 2-d)

Line integral $= \boxed{\int_a^b f(x(t),y(t))\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt} = \int_C f(x,y)\,ds$

In 3 variables

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\,|\vec{v}(t)|\,dt$$

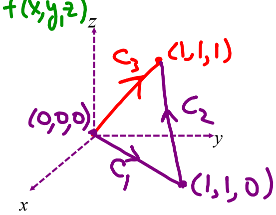
$$\boxed{= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$$

where $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\vec{r}\,'(t) = \vec{v}(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$

EX 1 The figure shows two different paths, $C_1 \cup C_2$ and C_3.

Find ① $\int_{C_3} \underbrace{(x-3y^2+z)}_{f(x,y,z)}ds$ and ② $\int_{C_1\cup C_2} \underbrace{(x-3y^2+z)}_{f(x,y,z)}ds$.



① C_3: $\vec{r}_3(t) = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$, $t \in [0,1]$

$\vec{r}_3\,'(t) = \langle 1,1,1\rangle$

$\|\vec{r}_3\,'(t)\| = \sqrt{3}$

$$\int_{C_3}(x-3y^2+z)ds = \int_0^1 (t-3t^2+t)\sqrt{3}\,dt$$ (remembering that $ds = \|\vec{r}\,'(t)\|dt$

$$= (t^2-t^3)\sqrt{3}\Big|_0^1 = (1-1)\sqrt{3} - 0 = 0$$

② C_1: $\vec{r}_1(t) = t\hat{\imath} + t\hat{\jmath}$, $t \in [0,1]$

$\vec{r}_1\,'(t) = \hat{\imath} + \hat{\jmath}$

$\|\vec{r}_1\,'(t)\| = \sqrt{2}$

C_2: $\vec{r}_2(t) = \hat{\imath} + \hat{\jmath} + t\hat{k}$, $t \in [0,1]$

$\vec{r}_2\,'(t) = 1\hat{k}$

$\Rightarrow \|\vec{r}_2\,'(t)\| = 1$

$$\int_{C_1\cup C_2}(x-3y^2+z)ds = \int_{C_1}(x-3y^2+z)ds + \int_{C_2}(x-3y^2+z)ds$$

$$= \int_0^1 (t-3t^2)\sqrt{2}\,dt + \int_0^1 (1-3+t)(1)\,dt$$

$$= \left(\sqrt{2}\frac{t^2}{2} - \frac{3\sqrt{2}t^3}{3}\right)\Big|_0^1 + \left(-2t+\frac{t^2}{2}\right)\Big|_0^1$$

$$= \frac{\sqrt{2}}{2} - \sqrt{2} - 0 + \left(-2+\frac{1}{2}\right) - 0$$

$$= -\frac{1}{2}\sqrt{2} - \frac{3}{2} = -\frac{1}{2}(\sqrt{2}+3) \neq 0$$

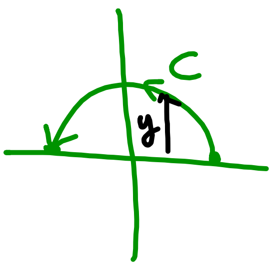
$$\Longrightarrow \int_{C_3} f(x,y,z)ds \neq \int_{C_1\cup C_2} f(x,y,z)ds$$

even though start & end pts are the same
(there are special fns where those integrals would be equal for all paths starting at one pt & ending together)

EX 2 A thin wire is bent in the shape of the semicircle

$$x = a\cos t, \quad t \in [0,\pi], \quad a > 0$$
$$y = a\sin t \qquad \text{(a fixed)}$$

If the density of the wire is proportional to the distance from the x-axis, find the mass of the wire.



density $\delta(x,y) = ky$ (k constant)

$= ka\sin t$

Δm = little bit of mass = density × length of small piece of wire

$\Rightarrow dm = \delta(x,y)\,ds$

total mass $= \displaystyle\int_0^{\pi} \delta(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = a\cos t$

$m = \displaystyle\int_0^{\pi} ka\sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$

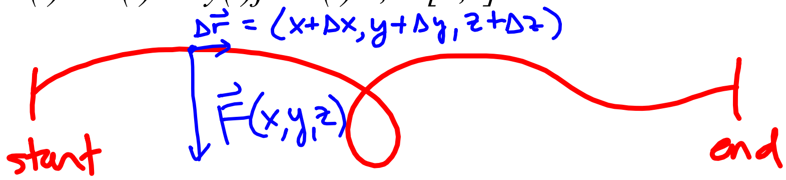
$= ka\displaystyle\int_0^{\pi} a\sin t\,dt$

$= ka^2\left(-\cos t\Big|_0^{\pi}\right) = -ka^2(-1-1) = 2ka^2$

Work

The goal is to calculate the work done by a vector field $\vec{F}(x,y,z)$ in moving an object along a curve C with parameterization.

C: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $t \in [a,b]$



The work done to move the object at (x,y,z) by a small vector, $\Delta\vec{r}$ is

$$\Delta W = \vec{F}(x,y,z) \cdot \Delta\vec{r}(x,y,z)$$

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Formula for calculating work

If $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

where $M = M(x,y,z)$
$N = N(x,y,z)$
$P = P(x,y,z)$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

then $W = \int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \left(\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right) dt$

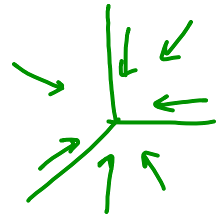
$$= \int_{t=a}^{t=b} \Big(M(x(t), y(t), z(t))\,dx + N(x(t), y(t), z(t))\,dy + P(x(t), y(t), z(t))\,dz \Big)$$

$$= \int_{t=a}^{t=b} \left(M\,dx + N\,dy + P\,dz \right)$$

EX 3 Find the work done by an inverse square law force field

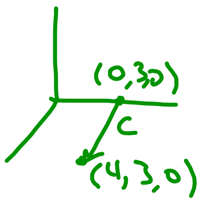
$$\vec{F}(x,y,z) = \frac{-c(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{x^2+y^2+z^2}}$$

in moving a particle along the straight line curve from $(0,3,0)$ to $(4,3,0)$.



Note: If $c > 0$, then the work done is negative.

A) find parameterization for the curve (line) from $(0,3,0)$ to $(4,3,0)$



line:

$x = 0 + 4t = 4t$
$y = 3 + 0t = 3$
$z = 0 + 0t = 0$
$t \in [0,1]$

$\vec{r}(t) = \langle 4t, 3, 0 \rangle$
$\vec{r}\,'(t) = \langle 4, 0, 0 \rangle$
$\|\vec{r}\,'(t)\| = 4$

B) $\vec{F}(t) = \vec{F}(x(t), y(t), z(t))$

$$= \frac{-c(4t\hat{i} + 3\hat{j} + 0\hat{k})}{\sqrt{16t^2+9+0}} = \frac{-c}{\sqrt{16t^2+9}}\langle 4t, 3, 0\rangle$$

C)

$$\Rightarrow W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\frac{-c}{\sqrt{16t^2+9}}\langle 4t,3,0\rangle \cdot \langle 4,0,0\rangle\right)dt$$

$$= \int_0^1 \frac{-c(16t)}{\sqrt{16t^2+9}}\,dt$$

$u = 16t^2+9$
$du = 32t\,dt$
$\frac{1}{2}du = 16t\,dt$

$t=0,\ u = 16(0)+9 = 9$
$t=1,\ u = 16+9 = 25$

$$= \int_9^{25} \frac{-c(\frac{1}{2})}{\sqrt{u}}\,du$$

$$= -\frac{c}{2}\int_9^{25} u^{-1/2}\,du$$

$$= -\frac{c}{2}\left(2u^{1/2}\right)\Big|_9^{25}$$

$$= -c(\sqrt{25} - \sqrt{9})$$

$$= \boxed{-2c}$$

EX 4 Evaluate $\int_C (2x+9z)ds$, where C is the curve given by

$x = t,\ y = t^2, z = t^3,\ \ t\in[0,1]$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$= \sqrt{1^2 + (2t)^2 + (3t^2)^2}\ dt$$

$$= \sqrt{9t^4+4t^2+1}\ dt$$

$$\int_C (2x+9z)\,ds$$

$$= \int_0^1 (2t+9t^3)\sqrt{9t^4+4t^2+1}\ dt$$

$u = 9t^4+4t^2+1$

$du = (36t^3+8t)dt$

$\frac{1}{4}du = (9t^3+2t)dt$

$$= \frac{1}{4}\int_1^{14} \sqrt{u}\ du = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\Big|_1^{14}\right)$$

$$= \frac{1}{6}\left(14^{3/2}-1\right)$$

EX 5 Evaluate $\int_C \left(y\,dx + x^2dy\right)$, where C is the curve given by

$x = 2t,\ y = t^2\text{-}1,\ t\in[0,2]$.

$dx = 2dt$

$dy = 2t\,dt$

$$\int_C (y\,dx + x^2dy) = \int_0^2 \left[(t^2-1)(2\,dt) + (2t)^2(2t)dt\right]$$

$$= \int_0^2 (2t^2-2+8t^3)\,dt$$

$$= \left(\frac{2t^3}{3} - 2t + \frac{8t^4}{4}\right)\Big|_0^2$$

$$= \left(\frac{2}{3}(8)-4+2(16)\right)-0 = \frac{16}{3}+28 = 33\frac{1}{3}$$