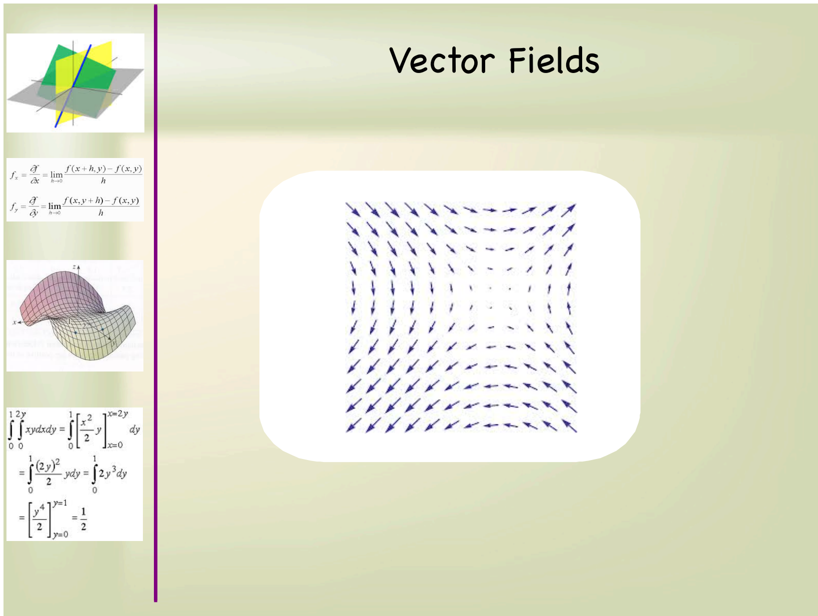

A Vector Field on a domain in space or in the plane is a function that assigns a vector to each point in the space.

EX 1 a) Attach a projectile's velocity vector to each point of its trajectory.

Domain: trajectory

Range: velocity field

b) Attach the gradient vector of a function to each point in the function's domain.

c) Attach a velocity vector to each point of a 3-D fluid flow.

EX 2 Plot the vector fields for each of these vector functions.

a) $\vec{F}(x,y) = x\hat{i} + y\hat{j}$

b) $\vec{F}(x,y) = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$

c) $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$

d) $\vec{F}(x,y) = \dfrac{-(x\hat{i} + y\hat{j})}{(x^2 + y^2)^{3/2}}$

<u>Scalar field</u>

$f: \mathbb{R}^3 \to \mathbb{R}$

$(x,y,z) \to f(x,y,z)$

<u>Vector field</u>

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

$(x,y,z) \to \langle M, N, P \rangle$

where $M = M(x,y,z)$

$N = N(x,y,z)$

$P = P(x,y,z)$

<u>Gradient of scalar field</u>

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

<u>Divergence of vector field</u>

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$ (scalar)

<u>Curl of vector field</u>

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \hat{j}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Think of ∇ as a vector-valued operator.

Then $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\begin{aligned} div\,\vec{F} &= \nabla \cdot \vec{F} \\ &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(M\hat{i} + N\hat{j} + P\hat{k}\right) \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \end{aligned}$$

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Note: If $\vec{F}$ is a velocity field for a fluid, then $div\,\vec{F}$ measures the tendency to diverge away from/toward a point.

$div\,\vec{F} > 0$ away

$div\,\vec{F} < 0$ toward

$curl\,\vec{F}$ - the direction about which the fluid rotates most rapidly.

$\|curl\,\vec{F}\|$ = speed of this rotation

EX 3 Let $\vec{F}(x,y,z) = e^x \cos y\hat{i} + e^x \sin y\hat{j} + z\hat{k}$

find

$\nabla \cdot \vec{F}$

$\nabla \times \vec{F}$

EX 4 Show that

a) $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$ for any $\vec{F}(x,y,z)$

b) $\nabla \times \left(\nabla f\right) = \vec{0}$ for any $f(x,y,z)$

A vector field is called <u>conservative</u> if

$\vec{F}(x,y,z) = \nabla f(x,y,z)$ for some $f : \mathbb{R}^3 \to \mathbb{R}$

then f is called the <u>potential function</u>.

EX 5 Let $\vec{F}(x,y,z) = \dfrac{-c\vec{r}}{\|\vec{r}\|^3}$

$$f(x,y,z) = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$$

show that

$$\vec{F} = \nabla f$$