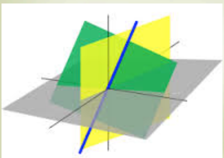
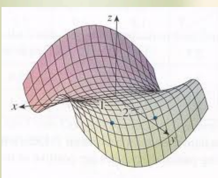


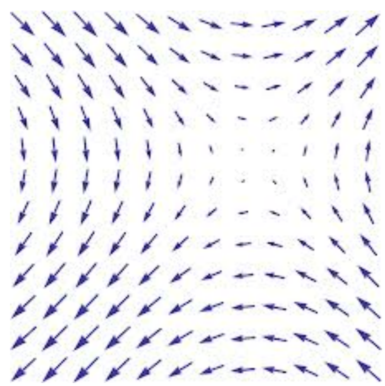
$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y}dy$$
$$= \int_0^1 \frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Vector Fields

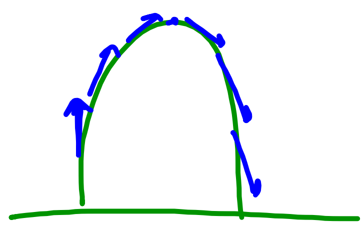


A Vector Field on a domain in space or in the plane is a function that assigns a vector to each point in the space. input: pt (x,y,z) output: vector

EX 1 a) Attach a projectile's velocity vector to each point of its trajectory.

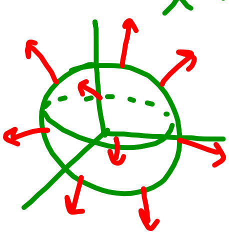
Domain: trajectory

Range: velocity field



b) Attach the gradient vector of a function to each point in the function's domain.

$x^2+y^2+z^2=a^2$ (sphere) $f(x,y,z)=x^2+y^2+z^2-a^2=0$



$\nabla f = \langle 2x, 2y, 2z \rangle$

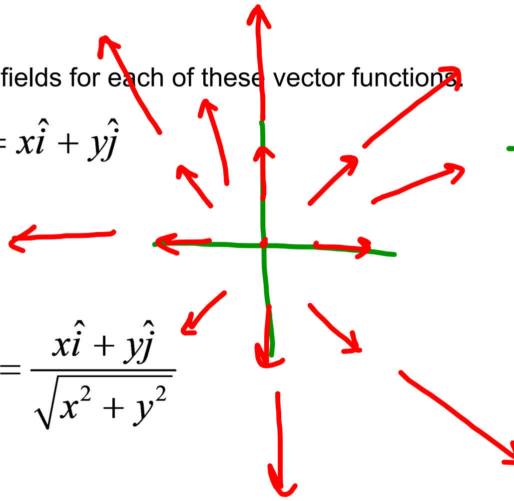
(pts away from origin)

c) Attach a velocity vector to each point of a 3-D fluid flow.

EX 2 Plot the vector fields for each of these vector functions.

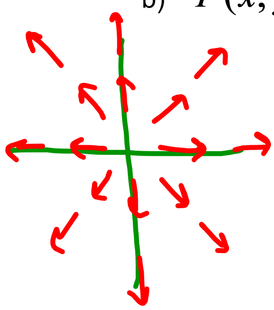
a) $\vec{F}(x,y) = x\hat{i} + y\hat{j}$

(radial vector field)



x	y	$\vec{F}$
0	0	$\langle 0,0\rangle = \vec{0}$
1	0	$\hat{i}$
0	1	$\hat{j}$
1	1	$\hat{i}+\hat{j}$
2	2	$2\hat{i}+2\hat{j}$

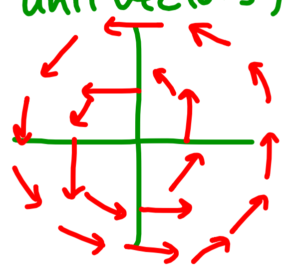
b) $\vec{F}(x,y) = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$



c) $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$

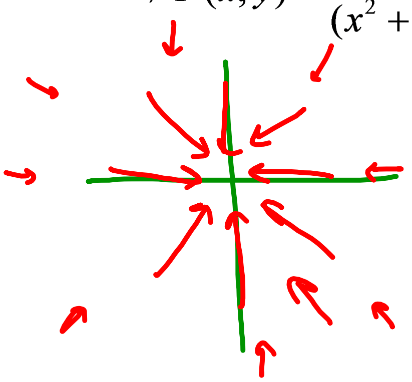
(called a circular vector field)

(all unit vectors)



x	y	$\vec{F}$
0	1	$-\hat{i}$
1	0	$\hat{j}$
-1	0	$-\hat{j}$
0	-1	$\hat{i}$

d) $\vec{F}(x,y) = \dfrac{-(x\hat{i} + y\hat{j})}{(x^2+y^2)^{3/2}}$



(as we move away from the origin, vectors get smaller)

(radial vector field)

Scalar field

$f: \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \to f(x, y, z)$

input: (x,y,z) pt

output: a number

Vector field

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\vec{F}$ maps $\mathbb{R}^3$ to $\mathbb{R}^3$

$(x, y, z) \to \langle M, N, P \rangle$

where $M = M(x,y,z)$

$N = N(x,y,z)$

$P = P(x,y,z)$

input: (x,y,z) pt

output: 3-d vector

Gradient of scalar field

input: 3-variable fn

output: 3-d vector

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Divergence of vector field

input: vector fn

output: scalar

$$\operatorname{div}(\vec{F}) \quad \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \quad \text{(scalar)}$$

$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

(∇ called nabla)

Curl of vector field

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \text{(operator)}$$

input: vector field

output: vector field

$$= \hat{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \hat{j}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Think of ∇ as a vector-valued operator.

remember:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Then $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$,

$M = M(x,y,z)$

$N = N(x,y,z)$

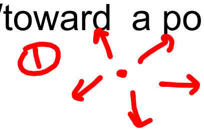
$P = P(x,y,z)$

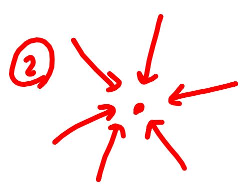
$$\begin{aligned} div\,\vec{F} &= \nabla \cdot \vec{F} \\ &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(M\hat{i} + N\hat{j} + P\hat{k}\right) \\ &= \boxed{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}} \end{aligned}$$

$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

<u>Note</u>: If $\vec{F}$ is a velocity field for a fluid, then $div\,\vec{F}$ measures the tendency to diverge away from/toward a point.

1 $div\,\vec{F} > 0$ away



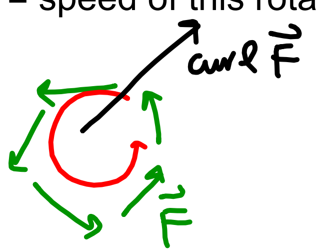


2 $div\,\vec{F} < 0$ toward

$curl\,\vec{F}$ - the direction about which the fluid rotates most rapidly.

$\|curl\,\vec{F}\|$ = speed of this rotation

(measures the tendency for vector field to rotate about a given pt)



EX 3 Let $\vec{F}(x,y,z) = e^x \cos y\hat{i} + e^x \sin y\hat{j} + z\hat{k}$

find

$\nabla \cdot \vec{F}$

$\nabla \times \vec{F}$

$$\nabla \cdot \vec{F} = \text{div}\,\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(e^x \cos y\,\hat{i} + e^x \sin y\,\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \sin y) + \frac{\partial}{\partial z}(z)$$

$$= e^x \cos y + e^x \cos y + 1$$

$$= 2e^x \cos y + 1$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & e^x \sin y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}(e^x \sin y)\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial}{\partial z}(e^x \cos y)\right) + \hat{k}\left(\frac{\partial}{\partial x}(e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y)\right)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}\left(e^x \sin y + e^x \sin y\right)$$

$$\nabla \times \vec{F} = 2e^x \sin y\,\hat{k}$$

EX 4 Show that

a) $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$ for any $\vec{F}(x,y,z)$

b) $\nabla \times \left(\nabla f\right) = \vec{0}$ for any $f(x,y,z)$

assuming that $\vec{F}(x,y,z) = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$

(a) $\nabla \cdot \left(\nabla \times \vec{F}\right) = \operatorname{div}\left(\operatorname{curl} \vec{F}\right) = 0$

$$\nabla \times \vec{F} = \hat{\imath}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial x} + \frac{\partial M}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y}$$

$= 0$ (as long as $\vec{F}$ is "nice")

i.e. $\dfrac{\partial^2 M}{\partial x \partial y} = \dfrac{\partial^2 M}{\partial y \partial x}$

(b) given fn $f(x,y,z)$

$\nabla \times \left(\nabla f\right) = \operatorname{curl}\left(\nabla f\right) = \vec{0}$ (claim)

$$\nabla f = f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}$$

$$\nabla \times \left(\nabla f\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \hat{\imath}\left(f_{zy} - f_{yz}\right) - \hat{\jmath}\left(f_{zx} - f_{xz}\right) + \hat{k}\left(f_{yx} - f_{xy}\right) = \vec{0}$$

A vector field is called <u>conservative</u> if

$$\vec{F}(x,y,z) = \nabla f(x,y,z) \quad \text{for some } f : \mathbb{R}^3 \to \mathbb{R}$$

(also known as gradient vector field)

$\hookrightarrow$ f is a function that maps from (x,y,z) to a #

then f is called the <u>potential function</u>.

EX 5 Let $\vec{F}(x,y,z) = \dfrac{-c\vec{r}}{\|\vec{r}\|^3}$

$$f(x,y,z) = \frac{c}{\sqrt{x^2+y^2+z^2}} = c\left(x^2+y^2+z^2\right)^{-1/2}$$

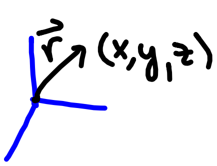
show that

$$\vec{F} = \nabla f$$

$$\nabla f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = -\tfrac{1}{2}c\left(x^2+y^2+z^2\right)^{-3/2}\left[2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}\right]$$

$$= \frac{-c}{\sqrt{\left(x^2+y^2+z^2\right)^3}}\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)$$

note: $\vec{r} = \langle x,y,z\rangle = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$



$$\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$$

$$\nabla f = \frac{-c(\vec{r})}{\|\vec{r}\|^3} = \vec{F} \checkmark$$