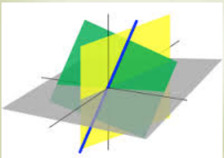
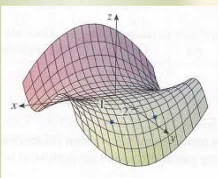


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

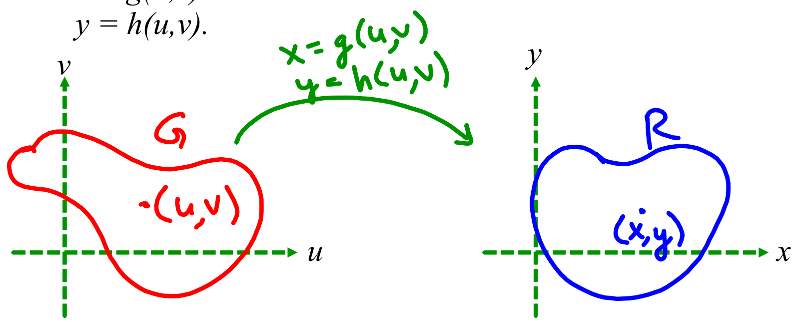
$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$

Change of Variables
(Jacobian Method)

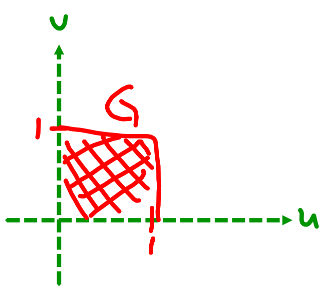
$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

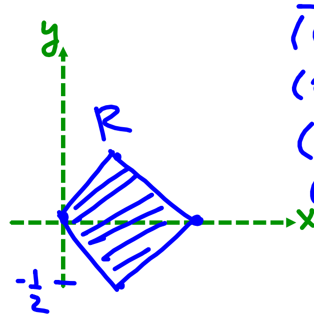
Transformations from a region G in the uv-plane to the region R in the xy-plane are done by equations of the form

$$x = g(u,v)$$
$$y = h(u,v).$$



Ex 1 $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$, G is the rectangle given by $0 \le u \le 1$, $0 \le v \le 1$.





(u,v)	(x,y)
(0,0)	(0,0)
(0,1)	$(\frac{1}{2}, -\frac{1}{2})$
(1,1)	(1, 0)
(1,0)	$(\frac{1}{2}, \frac{1}{2})$

From (u,v) to (x,y)

① $x = \dfrac{u+v}{2}$, ② $y = \dfrac{u-v}{2}$

From (x,y) to (u,v).

① $2x = u + v$
② $2y = u - v$

$2x + 2y = 2u$

Ⓐ $u = x + y$

① $v = 2x - u = 2x - (x+y)$

Ⓑ $v = x - y$

How is the integral of $f(x,y)$ over R related to the integral of $f(g(u,v),\, h(u,v))$ over G?

from (x,y) to (u,v)

$$\iint_R f(x,y)\,\boxed{dx\,dy} = \iint_G f(g(u,v),h(u,v))\,\boxed{|J(u,v)|\,du\,dv}$$

absolute value of Jacobian

where $$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

determinant

EX 2 For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, what is $J(r,\theta)$?

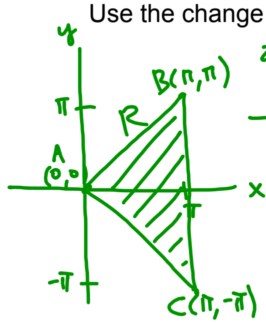
$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

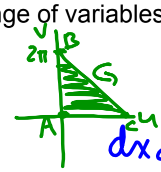
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

$\Longrightarrow$ $dx\,dy$ exchanged for $r\,dr\,d\theta$ for integration

EX 3 Evaluate $\iint_R \cos(x-y)\sin(x+y)\,dA$ where R is the triangle in the xy-plane with vertices at $(0,0)$, $(\pi,-\pi)$ and (π,π). Use the change of variables to $u = x - y$ and $v = x + y$.





$dA = dx\,dy$ (or $dy\,dx$)

$dx\,dy = |J(u,v)|\,du\,dv$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} - \left(\frac{-1}{4}\right) = \frac{1}{2}$$

$u = x - y \implies y = x - u$

$v = x + y$

$u + v = 2x$

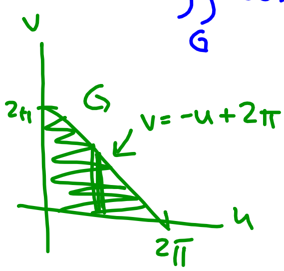
$$\boxed{x = \frac{u+v}{2}}$$

$y = \frac{u}{2} + \frac{v}{2} - u$

$$\boxed{y = \frac{v-u}{2}}$$

$$\iint_R \cos(x-y)\sin(x+y)\,dx\,dy$$

$$= \iint_G \cos(u)\sin(v)\left(\tfrac{1}{2}\,du\,dv\right)$$



$$= \int_0^{2\pi}\int_0^{2\pi-u} \frac{\cos u \sin v}{2}\,dv\,du$$

$$= \int_0^{2\pi} \frac{1}{2}\cos u\left(-\cos v\Big|_0^{2\pi-u}\right)du$$

$$= -\frac{1}{2}\int_0^{2\pi}\cos u\left(\cos(2\pi-u)-1\right)du$$

$$= -\frac{1}{2}\int_0^{2\pi}\cos u\cos(2\pi-u)-\cos u\,du$$

(because $\cos(2\pi-u)$ $= \cos(u-2\pi)$ even fn $= \cos u$ (due to periodicity))

$$= -\frac{1}{2}\int_0^{2\pi}\cos^2 u\,du + \frac{1}{2}\int_0^{2\pi}\cos u\,du$$

$$= -\frac{1}{2}\int_0^{2\pi}\frac{1+\cos(2u)}{2}\,du + \frac{1}{2}\cancel{\sin u\Big|_0^{2\pi}}$$

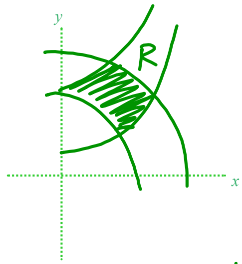
0

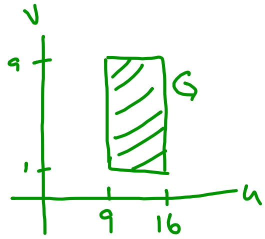
$$= \frac{-1}{4}\left(u + \frac{\sin(2u)}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{-1}{4}\left[(2\pi+0)-(0+0)\right] = \frac{-\pi}{2}$$

EX 4 Evaluate $\iint_R 5(x^2 + y^2)dx\,dy$ where R is the region in Quadrant I bounded by $\underbrace{x^2 + y^2 = 9,\ x^2 + y^2 = 16,}\ \underbrace{y^2 - x^2 = 1 \text{ and } y^2 - x^2 = 9}$.

Hint: Use $u = x^2 + y^2$ and $v = y^2 - x^2$ to transform R into a much nicer region (G).





$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

a bit of algebra:

$$\begin{cases} u = x^2 + y^2 \\ v = y^2 - x^2 \end{cases}$$

$$\begin{cases} x = \sqrt{\frac{u-v}{2}} \\ y = \sqrt{\frac{u+v}{2}} \end{cases}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{2(u-v)}} & \frac{-1}{2\sqrt{2(u-v)}} \\ \frac{1}{2\sqrt{2(u+v)}} & \frac{1}{2\sqrt{2(u+v)}} \end{vmatrix}$$

$$= \frac{1}{4\sqrt{u^2-v^2}}$$

$$\iint_R 5(x^2+y^2)\,dx\,dy$$

$$= \iint_G 5u\left(\frac{1}{4\sqrt{u^2-v^2}}\right)du\,dv$$

$$= \frac{5}{4}\int_1^9 \int_9^{16} \frac{u}{\sqrt{u^2-v^2}}\,du\,dv$$

$w = u^2 - v^2$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$u = 9,\ w = 9^2 - v^2$

$u = 16,\ w = 16^2 - v^2$

$$= \frac{5}{4}\int_1^9 \frac{1}{2}\int_{81-v^2}^{256-v^2} \frac{1}{\sqrt{w}}\,dw$$

$$= \frac{5}{8}\int_1^9 2\sqrt{w}\,\Big|_{81-v^2}^{256-v^2}\,dv$$

$$= \frac{5}{4}\int_1^9 \left(\sqrt{256-v^2} - \sqrt{81-v^2}\right)dv$$

$$= \frac{5}{4}\left(\frac{-256}{2}\cos^{-1}\left(\frac{v}{16}\right) + \frac{v}{2}\sqrt{256-v^2} - \left(-\frac{81}{2}\cos^{-1}\left(\frac{v}{9}\right) + \frac{v}{2}\sqrt{81-v^2}\right)\right)\Bigg|_1^9$$

Change of variables in 3 dimensions.

If $x = g(u,v,w)$
$y = h(u,v,w)$
$z = j(u,v,w)$

then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G f(g(u,v,w),h(u,v,w)\,j(u,v,w))\,|J(u,v,w)|\,du\,dv\,dw$$

where $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

EX 5 Let's check the Jacobian for spherical coordinates.

$$x = \rho\cos\theta\sin\varphi$$
$$y = \rho\sin\theta\sin\varphi$$
$$z = \rho\cos\varphi$$

$$J(\rho,\theta,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= \cos\varphi\left(-\rho^2\sin^2\theta\sin\varphi\cos\varphi - \rho^2\cos^2\theta\sin\varphi\cos\varphi\right)$$
$$+ -\rho\sin\varphi\left(\rho\cos^2\theta\sin^2\varphi - -\rho\sin^2\theta\sin^2\varphi\right)$$
$$= -\cos\varphi\,\rho^2\sin\varphi\cos\varphi(\sin^2\theta + \cos^2\theta)$$
$$- \rho^2\sin^3\varphi(\cos^2\theta + \sin^2\theta)$$
$$= -\rho^2\cos^2\varphi\sin\varphi - \rho^2\sin^3\varphi$$
$$= -\rho^2\sin\varphi(\cos^2\varphi + \sin^2\varphi)$$
$$= -\rho^2\sin\varphi$$

$$|J(\rho,\theta,\varphi)| = \rho^2\sin\varphi \implies dx\,dy\,dz = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$