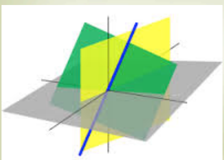
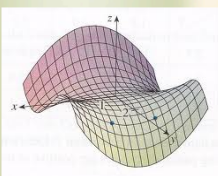


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

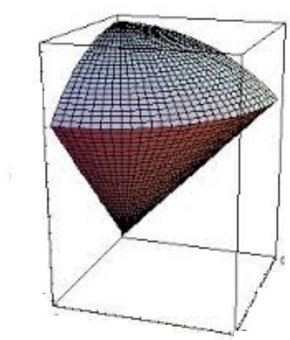


$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$

Triple Integrals in Cylindrical and Spherical Coordinates



Triple Integrals (Cylindrical and Spherical Coordinates)

$$\boxed{\iiint_S f(x,y,z)\,dV = \int_{\theta_1}^{\theta_{21}} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

$dV = dx\,dy\,dz$ (in rectangular coords)

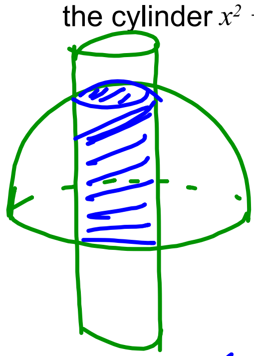
Note: Remember that in polar coordinates $dA = r\,dr\,d\theta$.

$$dV = dA\,dz = r\,dr\,d\theta\,dz$$

EX 1 Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 9$, below by the plane $z = 0$ and laterally by the cylinder $x^2 + y^2 = 4$. (Use cylindrical coordinates.)

sphere $r^2+z^2=9$

cylinder $r=2$



(cylinder w/ flat base and spherical top)

$$V = \iiint_S dV$$

S:
$0 \le z \le \sqrt{9-r^2}$
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left(z\Big|_0^{\sqrt{9-r^2}}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\sqrt{9-r^2}\,dr\,d\theta$$

$$= \left[\int_0^2 r\sqrt{9-r^2}\,dr\right]\left(\int_0^{2\pi} d\theta\right)$$

$u = 9-r^2$
$-\frac{1}{2}du = r\,dr$
$r=0,\ u=9$
$r=2,\ u=5$

$$= 2\pi\int_9^5 \left(-\tfrac{1}{2}\right)\sqrt{u}\,du$$

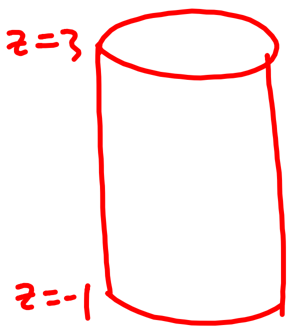
$$= -\pi\left(\frac{2}{3}u^{3/2}\right)\Big|_9^5$$

$$= -\frac{2\pi}{3}\left(5^{3/2}-27\right) = \boxed{\frac{2\pi}{3}\left(27-5^{3/2}\right)}$$

EX 2 Find $\iiint\limits_S f(x,y,z)\,dV$ for $f(x,y,z) = z^2\sqrt{x^2+y^2}$ and

$S = \{(x,y,z) \mid x^2 + y^2 \le 4,\ -1 \le z \le 3\}$.

$f = z^2 r$



S: $-1 \le z \le 3$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

(choose

$dV = r\,dr\,d\theta\,dz$

or $r\,dz\,d\theta\,dr$

or $r\,d\theta\,dr\,dz$, etc.)

$$\iiint\limits_S f\,dV = \int_0^{2\pi}\int_{-1}^{3}\int_0^{2} (z^2 r)\,r\,dr\,dz\,d\theta$$

$$= 2\pi \int_{-1}^{3} z^2\left(\frac{r^3}{3}\Big|_0^2\right)dz$$

$$= \frac{2\pi}{3}\int_{-1}^{3} z^2(2^3)\,dz = \frac{16\pi}{3}\left(\frac{z^3}{3}\Big|_{-1}^{3}\right)$$

$$= \frac{16\pi}{9}(27-(-1)) = \frac{28(16)\pi}{9}$$

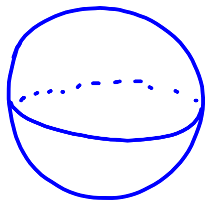
$$= \boxed{\frac{448\pi}{9}}$$

Spherical Coordinates

$$\boxed{\iiint_S f(x,y,z)\,dV = \int_{\phi_1}^{\phi_2}\int_{g_1(\phi)}^{g_2(\phi)}\int_{\psi_1(\theta,\phi)}^{\psi_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

notice: $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$ (can change order of $d\rho, d\theta, d\phi$)

EX 3 Find $\iiint_S f(x,y,z)dV$ for $f(x,y,z) = x^2 + y^2$ on $\underbrace{S = \{(x,y,z)\,|\, x^2 + y^2 + z^2 \le 1\}}_{\text{unit sphere}}$.



$$f = x^2 + y^2$$
$$= (x^2 + y^2 + z^2) - z^2$$
$$= \rho^2 - (\rho\cos\phi)^2$$
$$f = \rho^2(1 - \cos^2\phi)$$

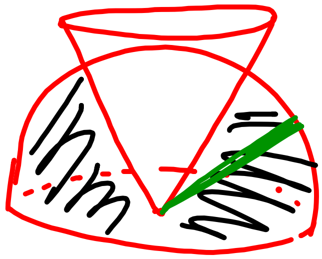
S: $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$$\iiint_S f\,dV = \int_0^{\pi}\int_0^1\int_0^{2\pi} \rho^2(1-\cos^2\phi)\rho^2\sin\phi\, d\theta\, d\rho\, d\phi$$
$$= 2\pi \int_0^{\pi}\int_0^1 \rho^4 \sin\phi(1-\cos^2\phi)\, d\rho\, d\phi$$
$$= 2\pi\left[\int_0^{\pi} \sin\phi(1-\cos^2\phi)\,d\phi\right]\left[\int_0^1 \rho^4\, d\rho\right]$$
$$= \frac{2\pi}{5}\int_0^{\pi}\left[\sin\phi - \sin\phi\cos^2\phi\right]d\phi$$
$$= \frac{2\pi}{5}\left(-\cos\phi\Big|_0^{\pi} + \int_1^{-1} u^2\, du\right)$$

$u = \cos\phi$
$du = -\sin\phi\, d\phi$
$\phi = 0,\ u = 1$
$\phi = \pi,\ u = -1$

$$= \frac{2\pi}{5}\left(-(-1-1) + \frac{u^3}{3}\Big|_1^{-1}\right)$$
$$= \frac{2\pi}{5}\left(2 + \frac{1}{3}(-1-1)\right)$$
$$= \frac{2\pi}{5}\left(\frac{4}{3}\right) = \boxed{\frac{8\pi}{15}}$$

EX 4 Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$, outside the cone, $z = \sqrt{x^2 + y^2}$, *and above the* xy-plane.



$$V = \iiint_S dV$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

sphere: $x^2+y^2+z^2=16$

$\Leftrightarrow \rho = 4$

cone: $z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$

$2z^2 = x^2+y^2+z^2$

$2\rho^2\cos^2\varphi = \rho^2$

$2\cos^2\varphi = 1$

$\cos\varphi = 1/\sqrt{2} = \sqrt{2}/2$

$\varphi = \pi/4$

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^4 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \left(\frac{\rho^3}{3}\Big|_0^4 \right) \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} 64 \sin\varphi \, d\varphi$$

$$= \frac{128\pi}{3} \left(-\cos\varphi \Big|_{\pi/4}^{\pi/2} \right)$$

$$= -\frac{128\pi}{3} \left(0 - \sqrt{2}/2 \right)$$

$$= \boxed{\frac{64\sqrt{2}\pi}{3}}$$