Double Integrals Over Rectangles, Iterated Integrals

\[ f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \]

\[ f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \]

\[
\int_0^2 \int_0^{2y} f(x,y) \, dy \, dx
\]

\[
= \int_0^2 \left[ \int_0^{2y} f(x,y) \, dy \right] \, dx
\]

\[
= \int_0^2 \left[ \frac{y^2}{2} \right] \, dx
\]

\[
= \left[ \frac{y^2}{2} \right]_{y=0}^{y=2} = \frac{4}{2} = 2
\]
EX 1 If \( f(x,y) = \begin{cases} -1 & 1 \leq x \leq 4, \ 0 \leq y < 1 \\ 2 & 1 \leq x \leq 4, \ 1 \leq y \leq 2 \end{cases} \)

find the signed volume between the \( z = f(x,y) \) surface and the xy-plane.

\[
V = V_{\text{box 1}} + V_{\text{box 2}}
\]

\[
V = 1(3)(-1) + 1(3)(2) = 3
\]
Definition (Double Integral)

Let \( z = f(x, y) \) be defined on a closed rectangle, \( R \).

If \( \lim_{|P| \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k \) exists, then \( f \) is integrable over \( R \)

and the double integral \( \iint_R f(x, y) \, dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k \)

Integrability Theorem

If \( f \) is continuous on the closed rectangle \( R \), then \( f \) is integrable on \( R \).

\[ f = f(x, y) \quad (R \text{ is in } xy\text{-plane}) \]

(i.e. I can find the signed volume of \( z = f(x, y) \) over \( R \))
EX 2 Let \( R = \{(x, y) \mid 0 \leq x \leq 6, \; 0 \leq y \leq 4\} \) and \( f(x, y) = 16 - y^2 \).

Partition \( R \) into 6 equal squares by lines \( x = 2, x = 4 \) and \( y = 2 \).

Approximate \( \iint_R f(x, y) \, dA \) as \( \sum_{k=1}^{6} f(\bar{x}_k, \bar{y}_k) \Delta A_k \)

where \((\bar{x}_k, \bar{y}_k)\) are centers of squares.

\[
z = 16 - y^2 \quad \text{(cylinder along x-axis)}
\]

\[
\sqrt{\approx \text{Volume of 6 rect. boxes}}
\]

base of each box \( \square^2 \Rightarrow \text{A base}=4 \)

ht of each box:

1. center: \((1, 3)\)
   
   ht of box over square \( \square = f(1, 3) \)

2. center: \((3, 3)\)
   
   ht of box = \( f(3, 3) \)

\[
\sqrt{\approx f(1, 3)(4) + f(3, 3)(4) + f(5, 3)(4) + f(1, 1)(4) + f(3, 1)(4) + f(5, 1)(4)}
\]

\[
\sqrt{\approx 4 \left[ (16 - 3^2) + (16 - 3^2) + (16 - 3^2) + (16 - 1^2) + (16 - 1^2) + (16 - 1^2) \right]}
\]

\[
= 4 \left( 21 + 45 \right) = 4 \times 66 = 264 \text{ units}^3
\]
Iterated Integrals

The total volume is the sum of many rectangular boxes and then we take the limit as the number of boxes goes to infinity to get the exact volume.

To find this volume, we can take thin "slab" cross-sections and add them up. Each slab has volume $A(y)\,dy$.

$$V = \int_c^d \left( \int_a^b f(x,y) \,dx \right) \,dy$$

or

$$V = \int_a^b \left( \int_c^d f(x,y) \,dy \right) \,dx$$

(work from "inside out")

(we can switch order of integrals easily here because we have fixed values of $x$ and $y$, i.e. $a, b, c, d$ are fixed.)
Properties of the Double Integral

A) It is a linear operator

1) \[ \iint_{R} kf(x,y) \, dA = k \iint_{R} f(x,y) \, dA \]

and

2) \[ \iint_{R} [f(x,y) + g(x,y)] \, dA = \iint_{R} f(x,y) \, dA + \iint_{R} g(x,y) \, dA \]

1) commutes w/ scalar multiplication

2) distributes through addition.

B) Additive on rectangles

\[ \iint_{R} f(x,y) \, dA = \iint_{R_1} f(x,y) \, dA + \iint_{R_2} f(x,y) \, dA \]

Where \( R_1 \) and \( R_2 \) overlap only on a line segment and comprise all of all \( R \).

C) If \( f(x,y) \leq g(x,y) \), then

\[ \iint_{R} f(x,y) \, dA \leq \iint_{R} g(x,y) \, dA \]

D) \[ \iint_{R} k \, dA = k \iint_{R} dA = k A(R) \]

\( k \) constant

\[ z = k \]
EX 3 Calculate \( \iiint_{R} f(x, y)\,dA \) where \( f(x, y) = 7 - y \)

\( R = \{(x, y) | 0 \leq x \leq 2, \quad 0 \leq y \leq 1\} \).

**Hint:** Sketch it and see if you recognize it.

\[
V = \iiint_{R} f(x, y)\,dA = \iint_{0}^{2} \int_{0}^{2} (7 - y)\,dx\,dy
\]

\[
= \int_{0}^{1} (7 - y) \left( \int_{0}^{2} \,dx \right)\,dy
\]

\[
= \int_{0}^{1} (7 - y) (2)\,dy
\]

\[
= \int_{0}^{1} (7 - y) (2)\,dy
\]

\[
= 2 \left( \frac{7y - y^2}{2} \right) \Bigg|_{0}^{1}
\]

\[
= 2 \left( \frac{7 - 1}{2} \right) - 0 = 13 \text{ units}^2
\]
Let's practice computing some double integrals.

EX 4 Evaluate \( \int_0^4 \left[ \int_{-1}^{1} (x^2 - 3y) \, dx \right] \, dy \)

\[
= \int_0^4 \left( \frac{x^3}{3} - 3yx \right) \bigg|_{-1}^{1} \, dy
\]

\[
= \int_0^4 \left( \left( \frac{8}{3} - 3y(2) \right) - \left( \frac{1}{3} - 3y(-1) \right) \right) \, dy
\]

\[
= \int_0^4 (3 - 9y) \, dy = \left( 3y - \frac{9y^2}{2} \right) \bigg|_0^4 = 12 - 9(8) - 0 = -60
\]

EX 5 \( \int_0^1 \int_0^1 \frac{y}{(xy + 1)^2} \, dx \, dy \)

\[
= \int_0^1 y \left( \int_0^1 \frac{1}{(xy + 1)^2} \, dx \right) \, dy
\]

\[
= \int_0^1 y \left[ \int_1^{xy+1} \frac{1}{u^2} \, du \right] \, dy
\]

\[
= \int_0^1 \frac{y}{y} \left[ \int_1^{y+1} u^{-2} \, du \right] \, dy
\]

\[
= \int_0^1 \left( \frac{y^{-1}}{y+1} \right) \, dy
\]

\[
= \int_0^1 \left( \frac{-1}{y+1} - \frac{-1}{y} \right) \, dy = \int_0^1 \frac{-1}{y+1} + 1 \, dy
\]

\[
= \left[ - \ln |y+1| + y \right]_0^1
\]

\[
= (-\ln 2 + 1) - (-\ln 1 + 0)
\]

\[
= 1 - \ln 2
\]
EX 6  \[ \int_R xy \sqrt{1+x^2} \, dA \]

\[ R = \{(x,y) \mid 0 \leq x \leq \sqrt{3}, 1 \leq y \leq 2\} \]

\[
= \int_0^{\sqrt{3}} \int_1^2 xy \sqrt{1+x^2} \, dy \, dx \\
= \int_0^{\sqrt{3}} x \sqrt{1+x^2} \left( \int_1^2 y \, dy \right) \, dx \\
= \int_0^{\sqrt{3}} x \sqrt{1+x^2} \left( \frac{y^2}{2} \right) \bigg|_1^2 \, dx \\
= \int_0^{\sqrt{3}} x \sqrt{1+x^2} \left( 2 - \frac{1}{2} \right) \, dx \\
= \frac{3}{2} \int_0^{\sqrt{3}} x \sqrt{1+x^2} \, dx \\
\begin{align*}
&= \frac{3}{2} \left[ \frac{1}{2} \sqrt{u} \right]_1^4 \quad \text{where} \quad u = 1 + x^2, \quad x = 0, \ u = 1 + 0^2 = 1 \\
&= \frac{3}{4} \int_1^4 \sqrt{u} \, du \\
&= \frac{3}{4} \left[ \frac{u^{3/2}}{3/2} \right]_1^4 \\
&= \frac{1}{2} \left( 4^{3/2} - 1^{3/2} \right) \\
&= \frac{1}{2} \left( 8 - 1 \right) = \sqrt{7/2}
\end{align*}

Note: If we have \( f(x,y) = g(x) h(y) \)

then \( \int_a^b \int_c^d f(x,y) \, dy \, dx \)

\[
= \int_a^b \left[ \int_c^d g(x) \, dy \right] \, dx \\
= \int_a^b \left[ g(x) \int_c^d h(y) \, dy \right] \, dx
\]

most of the time \( f(x,y) \neq h(y)g(x) \)
EX 7 Find the volume of the solid in the first octant enclosed by 
\[ z = 4-x^2 \text{ and } y = 2. \]

\[
V = \iint_R (4-x^2) \, dA
\]

domain: region

\[
\Rightarrow V = \int_0^2 \int_0^2 (4-x^2) \, dx \, dy
\]

\[
= \left( \int_0^2 dy \right) \left( \int_0^2 (4-x^2) \, dx \right)
\]

\[
= \left( y \bigg|_0^2 \right) \left( \left( 4x-x^3 \right) \bigg|_0^2 \right)
\]

\[
= (2-0)(8-\frac{8}{3})-0
\]

\[
= 2 \left( \frac{16}{3} \right) = \frac{32}{3}
\]