Tangent Planes

We already dealt with tangent planes to surfaces of form \( z = f(x,y) \).
Now, we will find tangent planes to surfaces of form \( F(x,y,z) = k \),
i.e. a surface represented by any equation in three variables.

Definition

Let \( F(x,y,z) = k \) be a surface, \( F \) differentiable at \( P(x_0,y_0,z_0) \) with
\( \nabla F(x_0,y_0,z_0) \neq \vec{0} \). Then the plane through \( P \) and perpendicular to
\( \nabla F(x_0,y_0,z_0) \) is called the tangent plane to the surface at \( P \).
Theorem
For surface $F(x,y,z)=k$, the equation of the tangent plane at $(x_0,y_0,z_0)$ is

$$\nabla F(x,y,z) \cdot (x-x_0,y-y_0,z-z_0) = 0$$

$$\Rightarrow$$

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$ 

EX 1 Find the equation of the tangent plane to $8x^2 + y^2 + 8z^2 = 16$ at \( \left( 1, 2, \frac{\sqrt{2}}{2} \right) \).
EX 2 Find parametric equations of the line that is tangent to the curve of intersection of these surfaces at the point \((1,2,2)\).

\[
\begin{align*}
f(x,y,z) &= 9x^2 + 4y^2 + 4z^2 - 41 = 0 \\
g(x,y,z) &= 2x^2 - y^2 + 3z^2 - 10 = 0
\end{align*}
\]

**Definition**

Let \(z = f(x,y)\), \(f\) is differentiable function, \(dx\) and \(dy\) (differentials) are variables. \(dz\) (also called total differential of \(f\)) is

\[
dz = df(x,y) = f_x(x,y)dx + f_y(x,y)dy = \nabla f \cdot (dx, dy).
\]

EX 3 Use differentials to approximate the change in \(z\) as \((x,y)\) moves from \(P\) to \(Q\). Also find \(\Delta z\).

\[z = x^2 - 5xy + y \quad P(2,3) \quad Q(2.03, 2.98)\]
EX 4 Use differentials to find the approximate amount of copper in the four sides and bottom of a rectangular copper tank that is 6 feet long, 4 feet wide and 3 feet deep inside, if the sheet-copper is 1/4 inch thick.

EX 5 A piece of cable (cylindrical) that measures 2 meters long with a radius of 2 centimeters is thought to have measurement error as large as 5 millimeters for each of the height and radius measurements. Estimate the error in the volume measurement.