Directional Derivatives

We know we can write

\[
\frac{df}{dx} = f_x(a,b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}
\]

\[
\frac{df}{dy} = f_y(a,b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}
\]

The partial derivatives measure the rate of change of the function at a point in the direction of the $x$-axis or $y$-axis. What about the rates of change in the other directions?

Definition

For any unit vector, $\hat{u} = \langle u_x, u_y \rangle$ let

\[
D_\hat{u} f(a,b) = \lim_{h \to 0} \frac{f(a + hu_x, b + hu_y) - f(a,b)}{h}
\]

If this limit exists, this is called the directional derivative of $f$ at the point $(a,b)$ in the direction of $\hat{u}$.

Theorem

Let $f$ be differentiable at the point $(a,b)$. Then $f$ has a directional derivative at $(a,b)$ in the direction of $\hat{u}$. $\hat{u} = u_x \hat{i} + u_y \hat{j}$ and

\[
D(\hat{u}) f(a,b) = \hat{u} \cdot \nabla f(a,b)
\]
EX 1 Find the directional derivative of \( f(x,y) \) at the point \((a,b)\) in the direction of \( \hat{u} \). (Note: \( \hat{u} \) may not be a unit vector.)

a) \( f(x,y) = y^2 \ln(x) \quad (a,b) = (1,4) \quad \hat{u} = \hat{i} - \hat{j} \)

b) \( f(x,y) = 2x^2 \sin y + xy \quad (a,b) = (1, \pi/2) \quad \hat{u} = 2\hat{i} + \hat{j} \)

Maximum Rate of Change

We know \( D_\theta f(a,b) = \hat{u} \cdot \nabla f(a,b) \)
\[= \|\hat{u}\| \|\nabla f(a,b)\| \cos \theta \]

What angle, \( \theta \), maximizes \( D_\theta f(a,b) \)?

**Theorem**

The function, \( z = f(x,y) \), increases most rapidly at \((a,b)\) in the direction of the gradient \( (\nabla f(a,b)) \) and decreases most rapidly in the opposite direction \( (-\nabla f(a,b)) \).

EX 2 For \( z = f(x,y) = x^2 + y^2 \), interpret gradient vector.
EX 3 Find a vector indicating the direction of most rapid increase of \( f(x,y) \) at the given point. Then find the rate of change in that direction.

a) \( f(x,y) = e^x \sin x \) at \((a,b) = (5\pi/6,0)\).

b) \( f(x,y) = x^2y - 2/(xy) \) at \((a,b) = (1,1)\)

EX 4 The temperature at \((x,y,z)\) of a ball centered at the origin is
\[
T = 100e^{-\left(x^2+y^2+z^2\right)}.
\]

Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.
One extra (cool) fact

**Theorem**
The gradient of \( z = f(x,y) \) \((w = f(x,y,z))\) at point \( P \) is perpendicular to the level curve (surface) of \( f \) through \( P \).

EX 5 Graph gradient vectors and level curves for

\[
z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{25}.
\]