Differentiability/Gradient

\[ f_x = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \]

\[ f_y = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} \]

\[
\int_{y_0}^{y_0+1} \int_{x_0}^{x_0+2} \rho(x, y) \, dx \, dy = \int_{y_0}^{y_0+1} \left( \int_{x_0}^{x_0+2} \rho(x, y) \, dx \right) \, dy \\
= \int_{y_0}^{y_0+1} \left( \frac{3y^2}{2} \right) \, dy = \frac{3}{2} \int_{y_0}^{y_0+1} y^2 \, dy \\
= \left[ \frac{y^3}{3} \right]_{y_0}^{y_0+1} = \frac{1}{2} 
\]
**Differentiability**
For a function of one variable, the derivative gives us the slope of the tangent line, and a function of one variable is differentiable if the derivative exists. For a function of two variables, the function is differentiable at a point if it has a tangent plane at that point. But existence of the first partial derivatives is not quite enough, unlike the one-variable case.

**Theorem**
If \( f(x,y) \) has continuous partial derivatives \( f_x(x,y) \) and \( f_y(x,y) \) on a disk \( D \) whose interior contains \( (a,b) \), then \( f(x,y) \) is differentiable at \( (a,b) \).

**Theorem**
If \( f \) is differentiable at \( (a,b) \), then \( f \) is continuous at \( (a,b) \).

\[ \text{differentiability } \Rightarrow \text{ continuity} \]
Gradient of $f$

$$\nabla f(p) = \nabla f(a,b) = \langle \hat{f}_x(a,b), \hat{f}_y(a,b) \rangle = \hat{f}_x(a,b)\hat{i} + \hat{f}_y(a,b)\hat{j}$$

for a function, $z = f(x,y)$.
(Note: This gradient lives in 2-D space, but it is the gradient of a function whose graph is 3-D.)

Gradient is a vector!!

Properties of Gradient Operator

$p$ is the input point $(a,b)$.

$$\begin{align*}
\nabla [f(p) + g(p)] &= \nabla f(p) + \nabla g(p) \\
\nabla [\alpha f(p)] &= \alpha \nabla f(p), \ \alpha \in \mathbb{R} \\
\nabla [f(p)g(p)] &= f(p)\nabla g(p) + \nabla f(p)g(p) \\
\text{("product rule")}
\end{align*}$$

Gradient is a linear operator
EX 1 Find the gradient of $f$.

a) $f(x,y) = x^3y - y^3$
\[
\nabla f = f_x \hat{i} + f_y \hat{j} = (3x^2y)\hat{i} + (x^3 - 3y^2)\hat{j}
\]

b) $f(x,y) = \sin^3(x^2y)$
\[
\nabla f = 3\sin^2(x^2y)\cos(x^2y)(2xy)\hat{i} + 3\sin^2(x^2y)\cos(x^2y)(x^3)\hat{j}
\]

c) $f(x,y,z) = xz \ln(x+y+z)$
\[
\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}
\]
\[
= (z \ln(x+y+z) + \frac{xz(1)}{x+y+z})\hat{i}
\]
\[
+ (\frac{xz(1)}{x+y+z})\hat{j} + (x \ln(x+y+z) + \frac{xz(1)}{x+y+z})\hat{k}
\]
Tangent Plane

Curves in 2-D

Remember the equation of the tangent line to a 2-D curve:

\[ y - f(a) = f'(a)(x-a) \]

\[ y = f(a) + f'(a)(x-a) \]

eqn of tangent line to curve \( y = f(x) \) (in 2-D)

Surfaces in 3-D

Find \( \vec{u} \) and \( \vec{v} \) (vectors in the tangent plane)

\[ \vec{n} = \vec{u} \times \vec{v} \]

\( \vec{u} = \text{no } y \text{-movement} \)

\[ = <1, 0, f_x(a, b)> \]

\( \vec{v} = \text{no } x \text{-movement} \)

\[ = <0, 1, f_y(a, b)> \]

\[ \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \hat{k}(-f_x(a, b)) - \hat{j}(f_y(a, b)) + \hat{i}(1) \]

\[ \vec{n} = <-f_x(a, b), -f_y(a, b), 1> \]

\[ \Rightarrow \text{eqn of plane } w/ \vec{n} \text{ normal vector} \]

and through pt \( (a, b, f(a, b)) \)

\[ <-f_x(a, b), -f_y(a, b)> \cdot <x-a, y-b, z-f(a, b)> = 0 \]

\[ -f_x(a, b)(x-a) - f_y(a, b)(y-b) + z - f(a, b) = 0 \]

\[ z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]

\[ \text{eqn of tangent plane to surface } z = f(xy) \text{ at } (a, b) \text{ input pt } (x, y, z) \]

or

\[ z = f(a, b) + \nabla f(a, b) \cdot <x-a, y-b> \]
EX 2 For $f(x,y) = x^3y + 3xy^2$, find the equation of the tangent plane at $(a,b) = (2,-2)$.

\[
\left< f_x(a,b), f_y(a,b) \right> \cdot \left< x-a, y-b \right> = z - f(a,b)
\]

\[
\nabla f(a,b) \cdot \left< x-a, y-b \right> = z - f(a,b)
\]

\[
z = f(a,b) + \nabla f(a,b) \cdot \left< x-a, y-b \right>
\]

\[
f_x = 3x^2y + 3y^2, \quad f_y = x^3 + 6xy
\]

\[
f(a,b) = f(2,-2) = 8(-2) + 3(2)(y) = 8
\]

Tangent plane: \[
z = 8 + \left< 3(2^3)(-2) + 3(2), 2^3 + 6(2)(-2) \right> \cdot \left< x-2, y+2 \right>
\]

\[
z = 8 + \left< -12, -16 \right> \cdot \left< x-2, y+2 \right>
\]

\[
z = 8 - 12(x-2) - 16(y+2)
\]

\[
z = -12x - 16y
\]

\[
12x + 16y + z = 0
\]
Ex 3 Find the equation of the tangent "hyperplane" to \( f(x,y,z) = w \) at the point \((a,b,c)\).

\[ f(x,y,z) = xyz + x^2 \quad (a,b,c) = (2,0,-3) \]

\[
w = f(a,b,c) + \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle
\]

\[
f(a,b,c) = f(2,0,-3) = 4
\]

\[
f_x = yz + 2x, \quad f_y = xz, \quad f_z = xy
\]

\[
f_x(2,0,-3) = 4, \quad f_y(2,0,-3) = -6, \quad f_z(2,0,-3) = 0
\]

\[\Rightarrow \nabla f(2,0,-3) = \langle 4, -6, 0 \rangle\]

Tangent hyperplane:

\[w = 4 + \langle 4, -6, 0 \rangle \cdot \langle x-2, y, z+3 \rangle\]

\[w = 4 + 4(x-2) - 6(y) + 0(z+3)\]

\[w = 4x - 6y - w = 4\]
Ex 4 Find all domain points \((x,y)\) at which the tangent plane to the graph of \(z = x^3\) is horizontal.

\[
\begin{align*}
2 &= f(x,y) = x^3 \\
\text{tangent plane horizontal} \\
\implies \text{normal of tangent plane} \langle 0, 0, 1 \rangle \\
\text{find the tangent plane to } z = f(x,y) \text{ at } (a,b) \\
2 &= a^3 + \nabla f(a,b) \cdot \langle x-a, y-b \rangle \\
2 &= a^3 + \langle 3a^2, 0 \rangle \cdot \langle x-a, y-b \rangle \\
2 &= a^3 + 3a^2(x-a) + 0 \\
2 &= 3a^2x - 3a^3 + a^3 \\
3a^2x - 2 &= 2a^3 \\
\text{normal vector is } \langle 3a^2, 0, -1 \rangle \\
\text{force } \langle 3a^2, 0, -1 \rangle &= c \langle 0, 0, 1 \rangle \\
\implies \text{let } c = -1, \quad -1 = -1 \checkmark (z\text{-component}) \\
3a^2 &= 0 \\
\implies a &= 0 \\
\implies \text{tangent plane is horizontal whenever } x = 0 \text{ and if } x=0, \quad f(x,y) = x^3 = 0 \text{ is true.} \\
\text{at pts of surface on y-axis tangent plane is horizontal.}
\end{align*}
\]