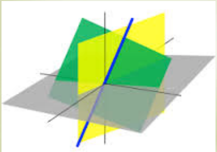
Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

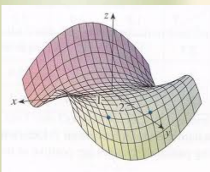
$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y\,\partial x^2} = f_{yxx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \frac{\partial^3 f}{\partial x^2\,\partial y} = f_{xxy}$$



$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



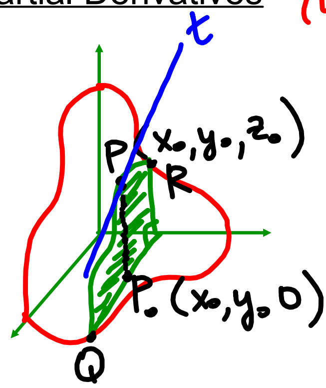
$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2}\,y\,dy = \int_0^1 2y^3\,dy$$

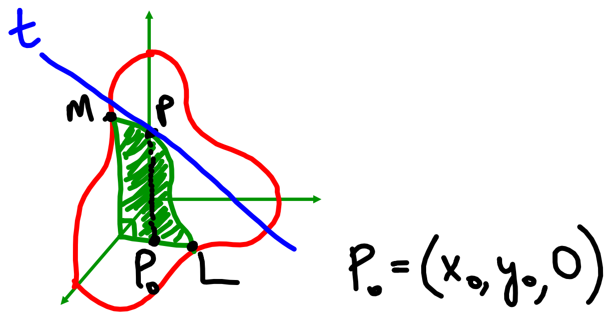
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Partial Derivatives

(a) cross-section || to xz-plane
(b) cross-section || to yz-plane



a (y is fixed)



b (x is fixed)

Consider the same surface cut by two different planes.

In **a** it is cut by $y = y_0$,

in **b** it is cut by $x = x_0$.

The curve of intersection in **a** goes through plane RPQ and in **b** through plane MPL.

Each of those curves has a tangent line associated with it at point P.

Each tangent line has a steepness associated with it and that should make us think about what? derivative!

Since our function is now a function of two variables (rather than one), we can only take the partial derivative with respect to one of the variables.

① $$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

"partial derivative of f with respect to x"

y remains constant

② $$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

(notation: wrt = with respect to)

x remains constant

note: to find f_x, treat y as a constant.
to find f_y, " x " " "

EX 1 Find $f_x(0,3)$ and $f_y(0,3)$ if $f(x,y)=3x^2y^2 + 4y^3 - 5$.

$f_x(x,y) = 3y^2(2x) + 0 - 0 = 6xy^2$

$f_y(x,y) = 3x^2(2y) + 12y^2 - 0 = 6x^2y + 12y^2$

$f_x(0,3) = 6(0)(3^2) = 0$

$f_y(0,3) = 6(0^2(3)) + 12(3^2) = 12(9) = 108$

Notation

If $z = f(x,y)$, then

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$ partial derivative of f with respect to x

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$ partial derivative of f with respect to y

EX 2 If $z = x^2y + cos(xy) - 2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

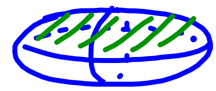
$$z = f(x,y) = x^2y + \cos(xy) - 2$$

$$\frac{\partial z}{\partial x} = f_x = y(2x) + -\sin(xy)(y)$$
$$= 2xy - y\sin(xy)$$

$$\frac{\partial z}{\partial y} = f_y = x^2(1) + -\sin(xy)(x)$$
$$= x^2 - x\sin(xy)$$

EX 3 Find the 'slope' of the tangent line to the curve of intersection of this surface $3z = \sqrt{36 - 9x^2 - 4y^2}$ (ellipsoid) top half

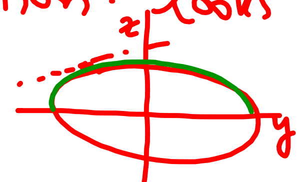
and the plane $x = 1$ (|| to yz-plane)



at the point $(1, -2, \sqrt{11}/3)$.

The 'slope' here refers to the change in z over the change in y.

the curve of intersection: looks like an ellipse



(in a plane || to yz-plane)

$$z = \frac{1}{3}\sqrt{36-9x^2-4y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{3}\left(\frac{1}{2}\right)(36-9x^2-4y^2)^{-1/2}(-8y)$$

$$= \frac{-4y}{3\sqrt{36-9x^2-4y^2}}$$

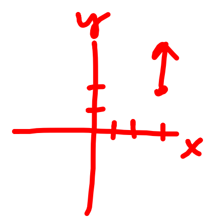
at $(1, -2, \sqrt{11}/3)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,-2,\sqrt{11}/3)} = \frac{-4(-2)}{3\sqrt{36-9(1)-4(4)}} = \frac{8}{3\sqrt{11}}$$

EX 4 The temperature in degrees celsius on a metal plate in the *xy*-plane is given by $T(x,y) = 4 + 2x^2 + y^3$. What is the rate of change of temperature with respect to distance (in feet) if we start moving from $(3,2)$ in the direction of the y-axis?

$\;x\;y$

want $\frac{\partial T}{\partial ?}$ (is it $\frac{\partial T}{\partial x}$ or $\frac{\partial T}{\partial y}$?)



(x remains constant)

$\Rightarrow$ want $\frac{\partial T}{\partial y}$.

$$\frac{\partial T}{\partial y} = 0 + 0 + 3y^2 = 3y^2$$

$$\Rightarrow \left.\frac{\partial T}{\partial y}\right|_{(3,2)} = 3(2^2) = 12\ {}^\circ\mathrm{C}/\mathrm{ft}$$

Higher Order Partial Derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

(notice the order of x and y)

EX 5 Find all four second partial derivatives for $f(x,y) = (x^3 + y^2)^5$.

$$f_x = 5(x^3+y^2)^4(3x^2) = 15x^2(x^3+y^2)^4$$

$$f_y = 5(x^3+y^2)^4(2y) = 10y(x^3+y^2)^4$$

$$f_{xy} = 15x^2(4)(x^3+y^2)^3(2y) = 120x^2y(x^3+y^2)^3$$

$$f_{yx} = 10y(4)(x^3+y^2)^3(3x^2) = 120yx^2(x^3+y^2)^3$$

(notice: $f_{xy} = f_{yx}$)

$$f_{xx} = 30x(x^3+y^2)^4 + 15x^2(4)(x^3+y^2)^3(3x^2)$$
$$= 30x(x^3+y^2)^3\left[x^3+y^2+6x^3\right] = 30x(x^3+y^2)^3\left[y^2+7x^3\right]$$

remember: $f_y = 10y(x^3+y^2)^4$

$$f_{yy} = 10(x^3+y^2)^4 + 10y(4)(x^3+y^2)^3(2y)$$
$$= 10(x^3+y^2)^3(x^3+y^2+8y^2) = 10(x^3+y^2)^3(x^3+9y^2)$$

EX 6 Find all four second partial derivatives for $f(x,y) = tan^{-1}(xy)$.

$$f_x = \frac{1}{1+(xy)^2}(y) = \frac{y}{1+x^2y^2} \text{ or } y(1+x^2y^2)^{-1}$$

$$f_y = \frac{1}{1+(xy)^2}(x) = \frac{x}{1+x^2y^2}$$

$$f_{xx} = \frac{-y}{(1+x^2y^2)^2}(2xy^2) = \frac{-2xy^3}{(1+x^2y^2)^2}$$

$$f_{yy} = \frac{-x}{(1+x^2y^2)^2}(2x^2y) = \frac{-2x^3y}{(1+x^2y^2)^2}$$

$$f_{xy} = \frac{(1+x^2y^2)(1) - y(2x^2y)}{(1+x^2y^2)^2} = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$$f_{yx} = \frac{(1+x^2y^2)(1) - x(2xy^2)}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$$f_{yx} = f_{xy}$$

fact: $f_{xy} = f_{yx}$ for all fns f that are "nice enough"

EX 7 For $f(x,y,z) = xy^2 - \frac{2x}{yz} + 3z^3x$, find f_x, f_y, f_z, f_{xz} *and* f_{yy}.

(this is a fn of 3 independent, input variables; it lives in 4-d space)

$f_x = y^2 - \frac{2}{yz} + 3z^3$

$f_y = 2xy - \frac{2x}{z}\left(\frac{-1}{y^2}\right) + 0 = 2xy + \frac{2x}{zy^2}$

$f_z = 0 - \frac{2x}{y}\left(\frac{-1}{z^2}\right) + 9z^2x = \frac{2x}{yz^2} + 9z^2x$

$f_{xz} = 0 - \frac{2}{y}\left(\frac{-1}{z^2}\right) + 9z^2$

$= \frac{2}{yz^2} + 9z^2$

$f_{yy} = 2x + \frac{2x}{z}\left(\frac{-2}{y^3}\right)$

$= 2x - \frac{4x}{zy^3}$