If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

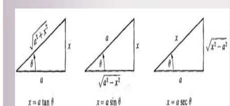
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

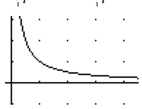
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Positive Series: Other Tests

Compare terms of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

Check the limit $\lim_{n\to\infty}\frac{a_n}{b_n}=L$

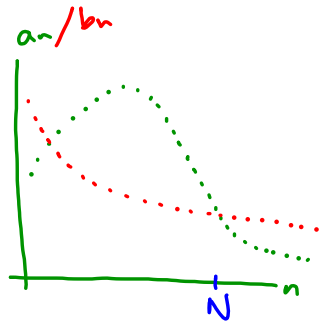
Check the ratio $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$

Two types of series may converge.

Geometric Series: $\sum_{n=1}^{\infty} ar^n$ converges if $|r|<1$.

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$.

<u>Ordinary Comparison Test</u> (OCT)



If $0 \le a_n \le b_n$ for every $n \ge N$

(i) If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

EX 1 Does $\sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5}$ converge or diverge?

(quick: p-series? no

geom. series? no

nth term test $\lim_{n\to\infty} \frac{3n+4}{4n^2-2n-5} = 0 \Rightarrow$ I know nothing!)

note: "develop your mathematical intuition"

as n gets large, the behavior of this series is like $\sum_{n=1}^{\infty} \frac{3n}{4n^2} = \sum_{n=1}^{\infty} \frac{3}{4}\left(\frac{1}{n}\right) = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n}$

this is a p-series, $p=1$ diverges)

use OCT:

$\sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5}$ $\quad 3n+4 > 3n \quad n=1,2,3,\dots$

also $4n^2-2n-5 < 4n^2$

$\Rightarrow \frac{1}{4n^2-2n-5} > \frac{1}{4n^2}$

$\Rightarrow \frac{3n+4}{4n^2-2n-5} > \frac{3n}{4n^2}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5} > \underbrace{\sum_{n=1}^{\infty} \frac{3}{4}\left(\frac{1}{n}\right)}_{=\infty}$

$\Rightarrow$ the series $\sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5}$ diverges (by OCT)

Limit Comparison Test (LCT) (this is among most frequently used)

Assume $a_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

1. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.
2. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

pf let $\varepsilon = \frac{L}{2}$. By defn of limits, we know $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ means that there exists an N such that when $n \geq N$

then $\left| \frac{a_n}{b_n} - L \right| < \varepsilon = \frac{L}{2}$

$\Leftrightarrow$

$$-\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2}$$

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2} \quad \text{(assuming } b_n > 0\text{)}$$

$$\frac{L}{2}(b_n) < a_n < \frac{3L}{2}(b_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{L}{2}(b_n) < \sum_{n=1}^{\infty} a_n < \underbrace{\sum_{n=1}^{\infty} \frac{3L}{2}(b_n)}_{= \frac{3L}{2} \sum_{n=1}^{\infty} b_n}$$

by OCT, if $\sum b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

And if $\sum a_n$ diverges, then $\sum b_n$ diverges.

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EX 2 Does this series converge or diverge?

$$\underset{n=1}{\frac{1}{1^2+1}}+\underset{n=2}{\frac{2}{2^2+1}}+\underset{n=3}{\frac{3}{3^2+1}}+\ldots = \sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+1}}_{a_n}$$

(quick: n^{th} term $\lim_{n\to\infty} \frac{n}{n^2+1} = 0 \Rightarrow$ we know nothing!

not geom. nor p-series)

Try LCT: We need to choose "b_n" that was referred to in LCT. Mostly, we'll choose $\sum b_n$ to be a p-series, because we know everything about p-series.

In our case, choose $b_n = \frac{1}{n}$. Then $\sum b_n$ diverges (p-series w/ $p=1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 < \infty$$

$\Rightarrow$ our series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges (along w/ $\sum b_n$)

Note:

Use LCT when you have "power of n" over a "power of n")

ex $\sum_{n=1}^{\infty} \frac{n^3+n^2-1}{n^4+n^5}$ ex $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^5-1}}{\sqrt{n^8+n}}$

EX 3 Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+1}$

(quick: $\lim_{n\to\infty} \frac{\sqrt{n+1}}{n^2+1} = 0$ we know nothing

not p-series

not geom series)

choose LCT because it's a power of n over a power of n.

choose $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ $\sum b_n$ converges, p-series w/ $p = 3/2 > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{n^2+1} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\ \sqrt{n^3}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^4+n^3}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^4}}{n^2} = 1 < \infty$$

$\Rightarrow$ by LCT, $\sum \frac{\sqrt{n+1}}{n^2+1}$ converges (along w/ $\sum b_n$)

Ratio Test (RT)

If $\sum a_n$ is a series of positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$,

then *i)* if $\rho < 1$, the series converges.

ii) if $\rho > 1$ or if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\infty$, the series diverges.

iii) if $\rho = 1$, then the test is inconclusive.

Note: Choose RT when you have a series with (a) exponentials and/or (b) factorials)

EX 4 Does this series converge or diverge? $3+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\ldots$

$n=1$ $n=2$ $n=3$ $n=4$

$$=\sum_{n=1}^{\infty}\frac{3^n}{n!}$$

(note: $3=\frac{3^1}{1!}$)

(note: $\lim_{n\to\infty}\frac{3^n}{n!}=0$)
know nothing

use RT (because there is an exponential and factorial)

$$\lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}=\lim_{n\to\infty}\frac{3\,\cancel{n!}}{(n+1)\cancel{n!}}$$

$$=\lim_{n\to\infty}\frac{3}{n+1}=0<1$$

$\Rightarrow$ by RT, our series converges.

EX 5 Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{n!}{5+n}$

quick: not p-series

not geom. series

n^{th} term test

$\lim_{n\to\infty} \frac{n!}{5+n} = \infty \Rightarrow$ series diverges.

RT:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{5+(n+1)} \cdot \frac{5+n}{n!}$$

$$= \lim_{n\to\infty} \frac{(5+n)(n+1)\cancel{n!}}{(6+n)\ \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n^2+6n+5}{n+6}$$

$$= \infty > 1 \Rightarrow \text{series diverges}$$

Conclusion (For positive series)

To test for convergence/divergence:

1. n^{th} term test for divergence.
2. Check for
   - (a) geometric series
   - or (b) p-series
3. Use
   - (a) LCT (if fraction of "power of n" over "power of n") choose $b_n = \frac{1}{n^p}$ where this represents the overall "essence" of a_n)
   - or (b) RT (if you have factorial of n and/or exponential)
4. try OCT
5. try integral test
6. try arguing w/ partial sums