If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

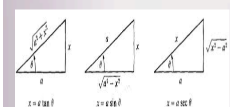
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

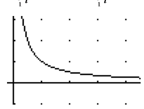
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

the product rule for differentiation

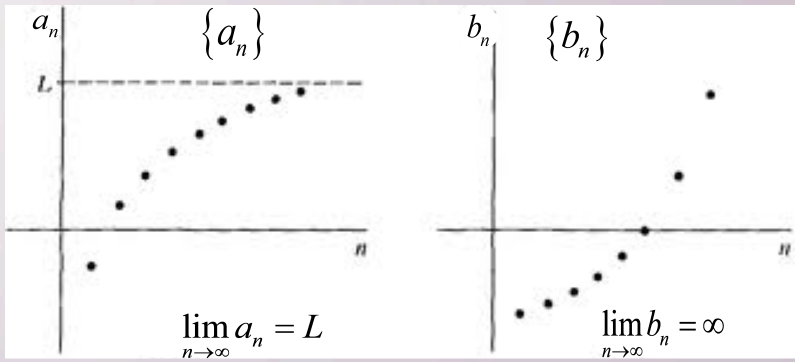
where it comes from: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

put into reverse $\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$

$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$

and then rearranged $\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$

Infinite Sequences



Definition

An infinite sequence is an ordered arrangement of real numbers.

$$a_1, a_2, a_3, a_4, \ldots$$

$$\{a_n\}_{n=1}^{\infty}$$

$$\{a_n\}$$

iteration (explicit) vs recursion (implicit)

$a_n = 5n - 3$, $n \geq 1$

(this is direct formula)

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 5 \qquad n \geq 2 \end{cases}$$

We can just write out the terms.

(arithmetic sequence) $2, 7, 12, 17, 22, \ldots$

Definition

Convergence

$\{a_n\}$ converges to L, written $\lim\limits_{n\to\infty} a_n = L$

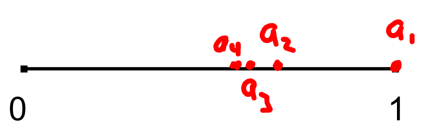
if for each positive ε there exists a corresponding positive N such that

$$n \geq N \Rightarrow |a_n - L| < \varepsilon \;.$$

(whenever n is big enough, a_n is really close to L)

If a sequence fails to converge to a finite L, then it diverges.

Example $a_n = \dfrac{n}{2n-1}$ $\quad n \geq 1$



n	a_n
1	1
2	$2/3$
3	$3/5$
4	$4/7$

$n = 1000$,

$a_{1000} = \dfrac{1000}{1999} \simeq \dfrac{1}{2}$

EX 1 Does $\{a_n\}$ converge? $a_n = \dfrac{5n^2 - 3n + 1}{2n^2 + 7}$

If so, what is the limit?

$$\lim_{n\to\infty} \frac{5n^2 - 3n + 1}{2n^2 + 7} = \lim_{n\to\infty} \frac{5n^2}{2n^2}$$

$\left(\frac{\infty}{\infty} \text{ case}\right)$

$$= \lim_{n\to\infty} \frac{5}{2} = \boxed{\frac{5}{2}}$$

yes, this sequence converges to 5/2

Properties of limits of sequences

Assume $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then

1) $\lim_{n\to\infty} k = k$

2) $\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad b_n \neq 0$

4) $\lim_{n\to\infty} k a_n = k \lim_{n\to\infty} a_n$

5) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

x is a continuous variable

n is a discrete variable

We can use l'Hopital's Rule.

EX 2 Determine if $\{a_n\}$ converges and if so, find $\lim_{n\to\infty} a_n$.

a) $a_n = \dfrac{\ln(1/n)}{\sqrt{2n}}$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{\sqrt{2n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{1/n}\left(\frac{-1}{n^2}\right)}{\cancel{\frac{1}{2}}(2n)^{-1/2}\cancel{(2)}} = \lim_{n\to\infty} \frac{-1/n}{\frac{1}{\sqrt{2n}}}$$

$\left(\frac{\infty}{\infty}\text{ case}\right)$

$$= \lim_{n\to\infty} \frac{-\sqrt{2n}}{n}$$

$$= \lim_{n\to\infty} \frac{-\sqrt{2}}{\sqrt{n}} = 0$$

b) $a_n = \dfrac{n^{100}}{e^n}$

$$\lim_{n\to\infty} \frac{n^{100}}{e^n} \overset{L}{=} \lim_{n\to\infty} \frac{100n^{99}}{e^n} \overset{L}{=} \lim_{n\to\infty} \frac{100(99)n^{98}}{e^n}$$

$\left(\frac{\infty}{\infty}\text{ case}\right)$

$$\overset{L}{=} \cdots \overset{L}{=} \lim_{n\to\infty} \frac{100!}{e^n} = 0$$

Squeeze Theorem

If $\{a_n\}$ and $\{c_n\}$ both converge to L and

$a_n \le b_n \le c_n$ for $n \ge K$ (some fixed integer),

then $\{b_n\}$ also converges to L.

EX 3 Determine if $\{a_n\}$ converges and if so, find $\lim\limits_{n\to\infty} a_n$.

$$\{a_n\} = e^{-n}\sin n$$

$$\lim_{n\to\infty} e^{-n}\sin(n) = \lim_{n\to\infty} \frac{\sin(n)}{e^n}$$

$$\lim_{n\to\infty} \frac{-1}{e^n} \le \lim_{n\to\infty} \frac{\sin(n)}{e^n} \le \lim_{n\to\infty} \frac{1}{e^n}$$

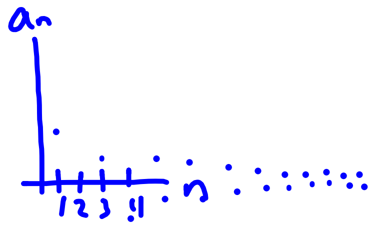
(because $\sin n \in [-1, 1]$)

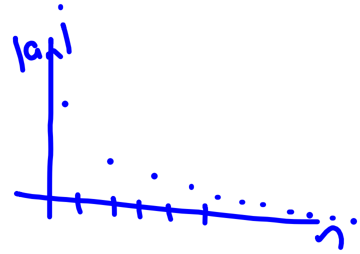
$$0 \le \lim_{n\to\infty} \frac{\sin(n)}{e^n} \le 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{e^n} = 0.$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.





EX 4 Show that if r is in this interval $(-1,1)$, then $\lim_{n\to\infty} r^n = 0$.

Pf Assume $r \neq 0$.

if $r \in (-1,1)$, then $|r| < 1$

$\Leftrightarrow \frac{1}{|r|} > 1$. So let $\frac{1}{|r|} = 1 + p$ for some $p > 0$.

$$\frac{1}{|r|^n} = (1+p)^n = 1 + np + \dots + p^n > np$$

$$\frac{1}{|r|^n} > np$$

$$|r|^n < \frac{1}{np}$$

we know $|r|^n \geq 0$.

$$0 \leq \lim_{n\to\infty} |r|^n \leq \lim_{n\to\infty} \frac{1}{np} = 0$$

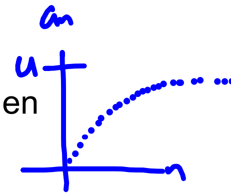
(p is fixed number > 0)

$$0 \leq \lim_{n\to\infty} |r|^n \leq 0$$

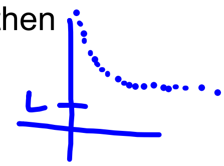
$$\Rightarrow \lim_{n\to\infty} |r|^n = 0 \Rightarrow \lim_{n\to\infty} r^n = 0 \ \#$$

Monotonic Sequence Theorem

If U is an upper bound for a nondecreasing sequence $\{a_n\}$,then the sequence converges to a limit A such that $A \le U$.



Also, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then the sequence converges to a limit B such that $B \ge L$.



EX 5 Write the first four terms for this sequence. Show that it converges.

$$\{a_n\} = \frac{n}{n+1}\left(2 - \frac{1}{n^2}\right)$$

n	a_n
1	$\frac{1}{2}\left(2-\frac{1}{1}\right) = \frac{1}{2}$
2	$\frac{2}{3}\left(2-\frac{1}{4}\right) = \frac{2}{3}\left(\frac{7}{4}\right) = \frac{7}{6}$
3	$\frac{3}{4}\left(2-\frac{1}{9}\right) = \frac{3}{4}\left(\frac{17}{9}\right) = \frac{17}{12}$
4	$\frac{4}{5}\left(2-\frac{1}{16}\right) = \frac{4}{5}\left(\frac{31}{16}\right) = \frac{31}{20}$

increasing sequence

$$\lim_{n\to\infty} \frac{n}{n+1}\left(2 - \frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2n}{(n+1)}\frac{(n)}{(n)} - \frac{\cancel{n}}{(n+1)n^{\cancel{2}}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2n^2-1}{n(n+1)}\right) = \lim_{n\to\infty} \frac{2n^2-1}{n^2+n}$$

$$= 2$$

$$\lim_{n\to\infty} \frac{n}{n+1}\left(2-\frac{1}{n^2}\right) = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\lim_{n\to\infty}\left(2-\frac{1}{n^2}\right)\right)$$

$$= 1 \cdot 2 = 2$$

Conclusion

- to determine if sequence converges or diverges,

  take limit as $n \to \infty$.

  If the limit is finite, it converges to that number.

  If the limit is $\pm\infty$ or DNE, then sequence diverges.