If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

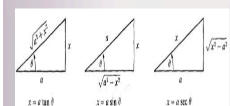
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

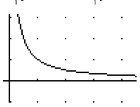
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

the product rule for differentiation

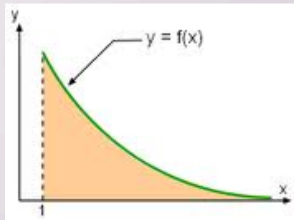
where it comes from: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

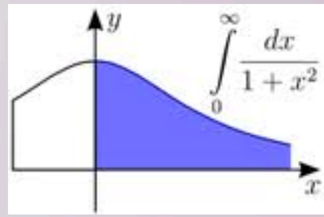
put into reverse $\int \frac{d}{dx}(uv) = \int (u\frac{dv}{dx} + v\frac{du}{dx})$

$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$

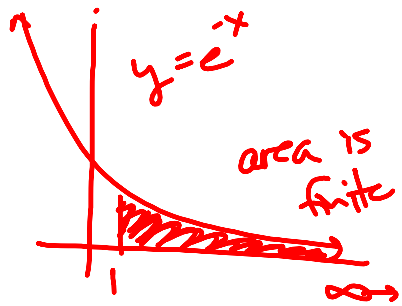
and then rearranged $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

Improper Integrals





<u>Improper Integral</u> It is like a definite integral except one or both of the limits of integration are $\pm\infty$.



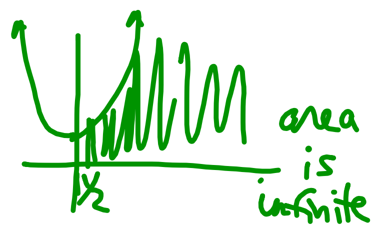
<u>Definition</u>

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx$$

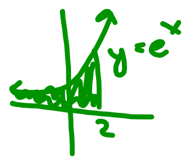
$$\int_{a}^{\infty} f(x)dx = \lim_{b\to\infty}\int_{a}^{b} f(x)dx$$

<u>converge</u> if the limit exists and is finite.

<u>diverge</u> if the limit does not exist (or goes to $\pm\infty$).



EX 1 $\int_{-\infty}^{2} e^{x}dx$



$$= \lim_{b\to-\infty}\int_{b}^{2} e^{x}dx$$

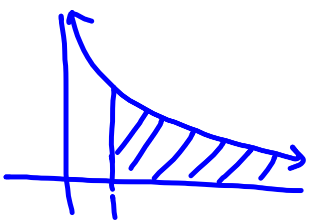
$$= \lim_{b\to-\infty}\left(e^{x}\Big|_{b}^{2}\right)$$

$$= e^{2} - \underbrace{\lim_{b\to-\infty} e^{b}}_{0} = e^{2}$$

(this integral converges)

EX 2 $\displaystyle\int_1^\infty \frac{1}{\sqrt{\pi x}}\,dx = \frac{1}{\sqrt{\pi}}\int_1^\infty x^{-1/2}\,dx$

$$= \lim_{a\to\infty} \frac{1}{\sqrt{\pi}}\int_1^a x^{-1/2}\,dx = \lim_{a\to\infty}\frac{1}{\sqrt{\pi}}\left(2x^{1/2}\Big|_1^a\right)$$



$$= \frac{2}{\sqrt{\pi}}\left(\underbrace{\lim_{a\to\infty}\sqrt{a}}_{\to\infty} - \sqrt{1}\right) \to \infty$$

diverges

EX 3 $\displaystyle\int_1^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{(1+x^2)^2}\,dx$

$u = 1+x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$x=1,\ u = 1+1^2 = 2$
$x\to\infty,\ u\to\infty$

$$= \lim_{b\to\infty}\frac{1}{2}\int_2^b \frac{1}{u^2}\,du$$

$$= \lim_{b\to\infty}\frac{1}{2}\left(-1\,u^{-1}\right)\Big|_2^b$$

$$= \lim_{b\to\infty}\frac{-1}{2u}\Big|_2^b$$

$$= \lim_{b\to\infty}\left(\underbrace{\cancel{\frac{-1}{2b}}}_{0}\right) - \left(\frac{-1}{2(2)}\right)$$

$$= \frac{1}{4}$$

Definition If $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$ converge,

then $\int_{-\infty}^{\infty} f(x)dx$ converges and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

otherwise, $\int_{-\infty}^{\infty} f(x)dx$ diverges.

EX 4 $\int_{1}^{\infty} \frac{1}{x^p} dx$

case 1: $p=1$

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx$$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b = \lim_{b\to\infty}(\ln b - \ln 1)$$

∞

diverges

case 2: $p>1$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b$$

$$= \frac{1}{-p+1}\left(\lim_{b\to\infty} b^{-p+1} - 1^{-p+1}\right)$$

$$= \frac{1}{-p+1}\left(\lim_{b\to\infty} \frac{1}{b^{p-1}} - 1\right)$$

0

$$= \frac{-1}{-p+1} = \frac{1}{p-1}$$

if $p>1$,
then
$-p<-1$
$-p+1<0$
$p-1>0$

Case 3: $p<1$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}dx \to \text{diverges}$$

(ex $p=-1$
$\frac{1}{x^{-1}} = x$
$\int_1^\infty x\,dx \to \infty$)

$$\boxed{\int_1^\infty \frac{1}{x^p}dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}}$$

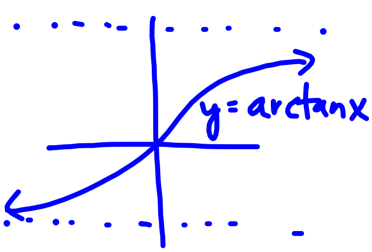
EX 5 $\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)} = \int_{-\infty}^{0} \frac{dx}{x^2+16} + \int_{0}^{\infty} \frac{dx}{x^2+16}$

$$= \lim_{b\to-\infty} \int_{b}^{0} \frac{dx}{x^2+16} + \lim_{a\to\infty} \int_{0}^{a} \frac{dx}{x^2+16}$$

$$= \lim_{b\to-\infty} \left(\frac{1}{4} \arctan\left(\frac{x}{4}\right) \Big|_{b}^{0} \right) + \lim_{a\to\infty} \left(\frac{1}{4} \arctan\left(\frac{x}{4}\right) \Big|_{0}^{a} \right)$$

$$= \frac{1}{4} \underbrace{\arctan(0)}_{0} - \lim_{b\to-\infty} \frac{1}{4} \arctan\left(\frac{b}{4}\right)$$

$$+ \lim_{a\to\infty} \frac{1}{4} \arctan\left(\frac{a}{4}\right) - \frac{1}{4} \underbrace{\arctan(0)}_{0}$$



$$= -\frac{1}{4}\left(\frac{-\pi}{2}\right) + \frac{1}{4}\left(\pi/2\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$$

EX 6 $\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+16}}\,dx = \int_{-\infty}^{0} \frac{x}{\sqrt{x^2+16}}\,dx + \int_{0}^{\infty} \frac{x}{\sqrt{x^2+16}}\,dx$

$u = x^2 + 16$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x \to -\infty,\ u \to \infty$

$x \to \infty,\ u \to \infty$

$x = 0,\ u = 0^2 + 16 = 16$

$$= \lim_{b\to\infty} \frac{1}{2}\int_{b}^{16} \frac{1}{\sqrt{u}}\,du + \lim_{a\to\infty} \frac{1}{2}\int_{16}^{a} \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{b\to\infty} \cancel{\tfrac{1}{2}}\left(\frac{u^{1/2}}{\cancel{1/2}}\right)\Bigg|_{b}^{16} + \lim_{a\to\infty} \cancel{\tfrac{1}{2}}\left(\frac{u^{1/2}}{\cancel{1/2}}\right)\Bigg|_{16}^{a}$$

$$= \sqrt{16} - \underbrace{\cancel{\lim_{b\to\infty}} \overset{\to\infty}{\sqrt{b}} + \cancel{\lim_{a\to\infty}} \overset{\to\infty}{\sqrt{a}}}_{\text{diverges}} - \sqrt{16}$$

Improper Integrals

- if area under $\overset{y=f(x)}{\text{curve}}$ is infinite, then $\int_a^{\infty} f(x)\,dx$ diverges.

- if $y=f(x)$ curve approaches the x-axis "fast enough," then the area under curve is finite, i.e. $\int_a^{\infty} f(x)\,dx < \infty$.

- for $\int_{-\infty}^{\infty} f(x)\,dx$, if either piece $\int_0^{\infty} f(x)\,dx$ or $\int_{-\infty}^{0} f(x)\,dx$ diverges, then the original integral diverges!