If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

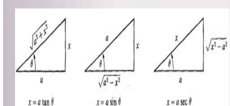
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

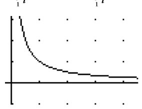
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt\cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Strategies for Integrating

$$\int\frac{f(x)}{g(x)}\,dx$$

Strategies for Integration

1. u-substitution
2. integration by parts
3. trigonometric integrals (use identities) (ex $\int \sin^3 x \, dx$) $\int \sin^2 x \cos^5 x \, dx$
4. rationalizing/trigonometric substitutions
5. partial fraction decomposition
6. integral tables
7. computer/calculator approximations (if we have definite integral)

EX 1 $\int \frac{1}{t-\sqrt{2t}} dt$

$u = \sqrt{2t} \Rightarrow u^2 = 2t \Leftrightarrow t = \frac{u^2}{2}$

$du = \cancel{\frac{1}{2}}(2t)^{-1/2}\cancel{(2)}\, dt$

$du = \frac{1}{\sqrt{2t}} dt$

$du = \frac{1}{u} dt$

$u\, du = dt$

$= \int \frac{1}{\frac{u^2}{2} - u}(u\, du)$

$= \int \frac{\cancel{u}}{\cancel{u}(\frac{1}{2}u - 1)} du$

$= \int \frac{1}{\frac{1}{2}u - 1} du$

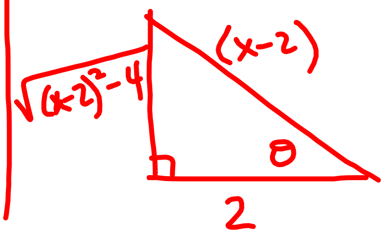
$= \frac{\ln\left|\frac{1}{2}u - 1\right|}{\frac{1}{2}} + C$

$= \boxed{2\ln\left|\frac{1}{2}\sqrt{2t} - 1\right| + C}$

EX 2 $\displaystyle\int \frac{\sqrt{x^2-4x}}{x-2}\,dx = \int \frac{\sqrt{(x-2)^2-4}}{(x-2)}\,dx$

complete the square:

$$x^2-4x = (x^2-4x+4)-4$$
$$= (x-2)^2-4$$



$$\sec\theta = \frac{x-2}{2}$$

$$x = 2+2\sec\theta$$
$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$= \int \sin\theta(2\sec\theta\tan\theta\,d\theta)$$

$$= 2\int \frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta}\right)d\theta = 2\int \tan^2\theta\,d\theta$$

$$= 2\int(\sec^2\theta-1)\,d\theta$$

$$= 2(\tan\theta-\theta)+C$$

$$= \cancel{2}\left(\frac{\sqrt{(x-2)^2-4}}{\cancel{2}}\right) - 2\sec^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= \boxed{\sqrt{x^2-4x} - 2\sec^{-1}\left(\frac{x-2}{2}\right)+C}$$

EX 3 $\int \frac{\sec h(\sqrt{x})}{\sqrt{x}} dx$ $= 2\int \operatorname{sech} u \, du$

$u = \sqrt{x}$

$du = \frac{1}{2} x^{-1/2} dx$

$2\, du = \frac{1}{\sqrt{x}} dx$

$= 2\left(\arctan\left|\sinh u\right|\right) + c$

$= \boxed{2 \arctan\left|\sinh \sqrt{x}\right| + c}$

EX 4 $\int x^2\sqrt{25-4x^2}\,dx$

In an integration table:

$\int u^2\sqrt{a^2-u^2}\,du$ (a constant)

$= \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}$

$+ \frac{a^4}{8}\sin^{-1}\left(\frac{u}{a}\right) + c$

$u = 2x \iff x = \frac{u}{2}$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$= \int \frac{u^2}{4}\sqrt{25-u^2}\left(\frac{1}{2}\right)du$

$= \frac{1}{8}\int u^2\sqrt{5^2-u^2}\,du$

$= \frac{1}{8}\left[\frac{u}{8}(2u^2-25)\sqrt{25-u^2} + \frac{5^4}{8}\sin^{-1}\left(\frac{u}{5}\right) + C\right]$

$= \frac{1}{8}\left(\frac{2x}{8}\left(2(2x)^2-25\right)\sqrt{25-(2x)^2} + \frac{625}{8}\sin^{-1}\left(\frac{2x}{5}\right)\right) + c$

$= \boxed{\frac{x}{32}(8x^2-25)\sqrt{25-4x^2} + \frac{625}{64}\sin^{-1}\left(\frac{2x}{5}\right) + c}$