Definition: Continuity at a Point

Let $f$ be defined on an open interval containing $c$. We say that $f$ is continuous at $c$ if

$$\lim_{x \to c} f(x) = f(c)$$

This indicates three things:

1. The function is defined at $x = c$.
2. The limit exists at $x = c$.
3. The limit at $x = c$ needs to be exactly the value of the function at $x = c$.

Three examples:
6 Continuity

Continuous Functions

a) All polynomial functions are continuous everywhere.
b) All rational functions are continuous over their domain.
c) The absolute value function is continuous everywhere.
d) \( f(x) = \sqrt[n]{x} \) is continuous for all real numbers if \( n \) is odd.
e) \( f(x) = \sqrt{x} \) is continuous for all non-negative real numbers if \( n \) is even.
f) The sine and cosine functions are continuous over all real numbers.
g) The cotangent, cosecant, secant and tangent functions are continuous over their domain.

More continuous functions
If \( f(x) \) and \( g(x) \) are continuous at \( x = c \), then so are

\[
\begin{align*}
    kf(x), & \quad (f \pm g)(x), & \quad (f \cdot g)(x), & \quad \frac{f}{g}(x), \quad (g(x) \neq 0), \\
    f^n(x), & \quad \sqrt[n]{f(x)}, & \quad (f(c) > 0 \text{ if } n \text{ is even}).
\end{align*}
\]

EX 1 State where these functions are continuous.

a) \( f(x) = x^2 - 9 \)

b) \( g(x) = \sqrt{x} - 5 \)

c) \( h(x) = \frac{21 - 7x}{x - 3} \)

d) \( p(x) = \begin{cases} 
    7 - 3x & x \leq 3 \\
    -2 & x > 3 
\end{cases} \)
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**Composite Limit Theorem**

If \( \lim_{x \to c} g(x) = L \) and \( f \) is continuous at \( L \), then

\[
\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(L)
\]

Ex 2  At what points are the following functions continuous?

a) \( h(x) = \frac{1}{\sqrt{4 + x^2}} \)

b) \( g(t) = |t - 2| \)

Ex 3  If \( f(x) = \frac{x^2 - 49}{x - 7} \), how do we need to complete the definition for this to be continuous everywhere?

**Intermediate Value Theorem**

\( f \) is a function defined on \([a,b]\) and \( \omega \) is a number between \( f(a) \) and \( f(b) \).

If \( f \) is continuous on \([a,b]\), then there exists at least one number, \( c \), \( (a < c < b) \) such that \( f(c) = \omega \).
Use interval notation to state all values for which this function is continuous.