Limits At Infinity, Infinite Limits
4B Limits at Infinity

Definition: (Limit as $x \to \infty$)

$f(x)$ is defined on $[c, \infty)$, $c \in \mathbb{R}$ (or $(-\infty, c]$)

We say that if for every $\varepsilon > 0$, there is a corresponding number, $m$ such that

$x > m \Rightarrow |f(x) - L| < \varepsilon$, then $\lim_{x \to \infty} f(x) = L$

as $x$ gets bigger, the y-value $(f(x))$ gets closer to $L$. 

\[\lim_{x \to -\infty} f(x) = L\]
4B Limits at Infinity

EX 1  Intuitively (looking at the graph) determine these limits.

\[
\lim_{x \to \infty} f(x) = 1
\]

\[
\lim_{x \to -\infty} f(x) = 0
\]

EX 2  Show that if \( n \) is a positive integer, then \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \).

Let \( \varepsilon > 0 \) be given. Choose \( M = \left( \frac{1}{\varepsilon} \right)^{\frac{1}{n}} = \sqrt[n]{\frac{1}{\varepsilon}} \).

Then \( x > m \Rightarrow \frac{x^n}{x^m} > \frac{m^n}{x^m} \)

\[
\Rightarrow \quad \frac{1}{x^n} > \frac{1}{m^n}
\]

So \( \left| \frac{1}{x^n} - 0 \right| = \left| \frac{1}{x^n} \right| < \frac{1}{m^n} = \left( \sqrt[n]{\frac{1}{\varepsilon}} \right)^n = \frac{1}{\varepsilon} = \varepsilon \)

i.e. \( \left| \frac{1}{x^n} - 0 \right| < \varepsilon \).

\Rightarrow \text{by defn} \quad \lim_{x \to \infty} \frac{1}{x^n} = 0 \quad (n \in \mathbb{Z}^+) \quad \Rightarrow \text{element of positive integers}
4B Limits at Infinity

EX 3
\[
\lim_{{x \to \infty}} \frac{2x + 3}{x^2 + 1} = \lim_{{x \to \infty}} \left( \frac{2x + 3}{x^2 + 1} \cdot \frac{x^2}{x^2} \right)
\]

Highest degree term \( \cdot \) \( x \)
\[= \lim_{{x \to \infty}} \frac{2x}{x} = \lim_{{x \to \infty}} \frac{2}{1} = 0
\]

EX 4
\[
\lim_{{x \to \infty}} \frac{2x^3 + 3x^2 + 5x + 7}{x^3 + 7} = \lim_{{x \to \infty}} \frac{3x^4}{x^3}
\]

= \lim_{{x \to \infty}} 3x \to \infty

Note of warning:
This shortcut works only when we have limit as \( x \to \pm \infty \)

EX 5
\[
\lim_{{x \to \infty}} \frac{2x^2 + 5x - 1}{x^2 + 3x} = \lim_{{x \to \infty}} \frac{2x^2}{x^2}
\]

= \lim_{{x \to \infty}} 2 = 2

Limits of Rational Fns as \( x \to \pm \infty \)

1. If degree of \( n(x) \) < degree of \( d(x) \), then limit goes to 0.

2. If degree of \( n(x) \) ≥ degree of \( d(x) \), then limit goes to \( \pm \infty \) (have to look at coefficients of leading terms).

3. If degree of \( n(x) = \) degree of \( d(x) \), then limit is quotient of the leading coefficients.
Definition: (Infinite limit )

We say $\lim_{x \to c} f(x) = \infty$ if for every positive number, $m$

there is a corresponding $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) > m$

(x→c⁺: means x is going to c from the right)
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**EX 6** Determine these limits looking at this graph of \( f(x) = \frac{1}{x-1} \).

\[
\lim_{{x \to \infty}} f(x) = \infty
\]

\[
\lim_{{x \to -\infty}} f(x) = -\infty
\]

\[
\lim_{{x \to 1^+}} f(x) = \infty
\]

\[
\lim_{{x \to 1^-}} f(x) = -\infty
\]

**Ex 7** Find the horizontal and vertical asymptotes for this function, then write a few limit statements including \( \infty \).

\[
f(x) = \frac{-2x}{x + 3}
\]

**VA:** of form \( x = c \), \( c \) is a constant

(try graph of \( y = f(x) \) can never cross or touch the VA)

find it by looking at domain restrictions

\( f(x) \) has problem at \( x = -3 \) (make den. \( \neq 0 \))

\( \Rightarrow \) VA: \( x = -3 \) (domain: \( x \in \mathbb{R}, x \neq -3 \))

\[
\lim_{{x \to -3^+}} \frac{-2x}{x+3} = \infty
\]

\[
\lim_{{x \to -3^-}} \frac{-2x}{x+3} = -\infty
\]

**HA:** horiz. line that fn approaches, eventually (as \( x \) gets huge)

\[
y = \lim_{{x \to \pm \infty}} f(x)
\]

\[
y = \lim_{{x \to \infty}} \frac{-2x}{x+3} = \lim_{{x \to -\infty}} \frac{-2x}{x} = -2 \Rightarrow HA: y = -2
\]
Ex 8  
a) Find the vertical and horizontal asymptotes for this function.

\[ f(x) = \frac{2x}{\sqrt{x^2 + 5}} \]

VA: (domain restrictions) none

HA: \( \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + 5}} \)

\[ = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{2x}{x} \]

\[ = \lim_{x \to \infty} 2 = 2 \]

\[ \Rightarrow \text{HA: } y=2 \]

Note: \( \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \)

\[ \text{ex. } \sqrt{5^2} = 5 \]

d) Determine these limits:

\[ \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2}} \]

\[ = \lim_{x \to -\infty} \frac{2}{-1} = -2 \]

\[ \Rightarrow \text{HA: } y=-2 \]

Note: HA(s) describe behavior of y-value as x gets huge!!

(we can cross the HA as many times as fn requires when x is not "huge")
Determine these limits:

\[
\lim_{x \to 0^-} f(x) = 2
\]

\[
\lim_{x \to 0^+} f(x) = \infty
\]

\[
\lim_{x \to \infty} f(x) = 2
\]

\[
\lim_{x \to 0} f(x) = -\infty
\]

\[
\lim_{x \to 0} f(x) \text{ DNE}
\]