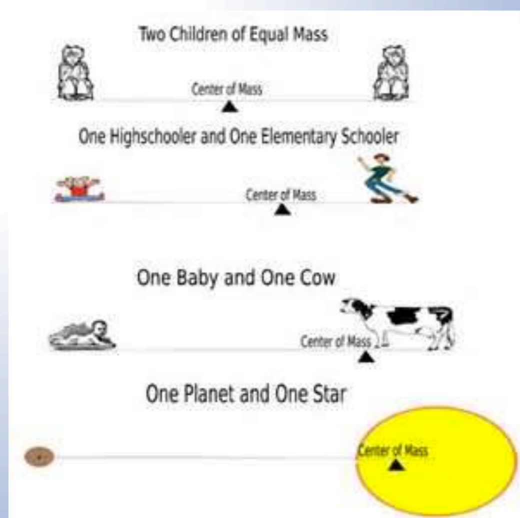
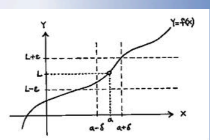
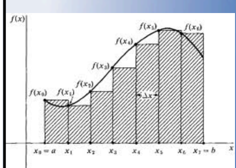
Moments, Center of Mass





$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

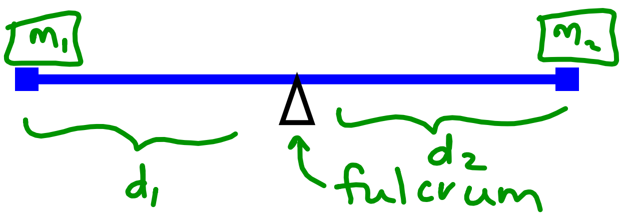
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$



$$\lim_{max\, \Delta x_i \to 0} \sum_1^n f(x_i) \Delta x_i = \int_a^b f(x) dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

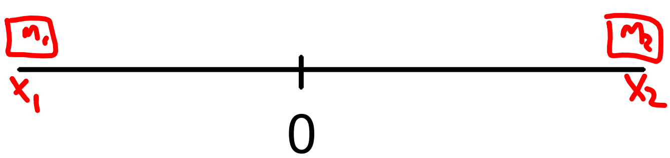
"teeter totter"



It stays balanced if

$m_1 d_1 = m_2 d_2$

Let's put fulcrum at origin on an x-axis.



for balance

$m_1 x_1 + m_2 x_2 = 0$

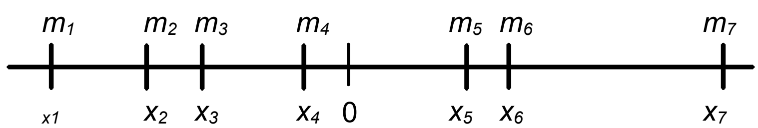
★ x_1, x_2 are directed distances

The moment of a particle with respect to a point is the product of mass (m) of the particle with its directed distance (x) from a point. This measures the tendency to produce a rotation about that point.

Total moment (M) for a bunch of masses = $\sum_{i=1}^{n} x_i m_i$

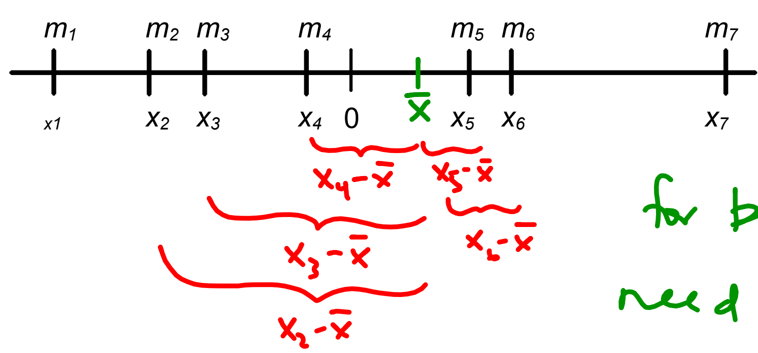
x_i = distance from 0 to particle i

m_i = mass of particle i



Where does the fulcrum need to be placed to balance? Let's call it $\bar{x}$.

(assume we have n particles) on x-axis



for balance, we need

$$(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 + (x_3-\bar{x})m_3 + \dots + (x_n-\bar{x})m_n = 0$$

we are looking for formula for $\bar{x}$.

$$x_1 m_1 + x_2 m_2 + x_3 m_3 + \dots + x_n m_n = \bar{x} m_1 + \bar{x} m_2 + \bar{x} m_3 + \dots + \bar{x} m_n$$

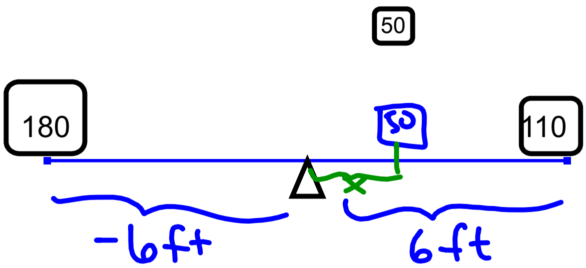
$$\sum_{i=1}^{n} x_i m_i = \bar{x} \sum_{i=1}^{n} m_i$$

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}}$$

location of fulcrum for balance in a 1-d discrete case

EX 1

John and Mary, weighing 180 lbs and 110 lbs respectively, sit at opposite ends of a 12-ft teeter-totter with the fulcrum in the middle. Where should their 50-lb son sit in order for the board to balance?



for balance we need

$$180(-6) + 50x + 110(6) = 0$$

$$50x = 420$$

$$x = 8.4$$

$\Rightarrow$ this says son must be 8.4 ft to right of fulcrum; that means he is not on teeter totter

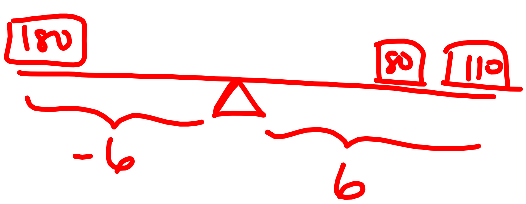
i.e. there's no way to get balance on this teeter totter

let's say son weighs 80 lbs instead.

we get: $180(-6) + 80x + 110(6) = 0$

$$80x = 420$$

$$x = \frac{42}{8} = \boxed{5.25 \text{ ft}}$$



For a continuous mass distribution along the line (like on a wire):

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}}$$

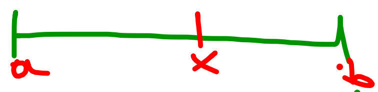
since total mass is $\int_a^b \delta(x)\,dx$

& moment is $\int_a^b \delta(x)\,x\,dx$

($\delta(x)$ is density fn)

units: mass/distance

same as discrete case,

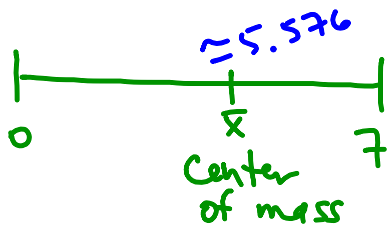


except we swap finite sums for integrals

$$\underbrace{\delta(x)\,dx}_{\text{a bit of mass}} \Rightarrow \text{total mass} = \int_a^b \delta(x)\,dx$$

EX 2

A straight wire 7 units long has density $\delta(x) = 1+x^3$ at a point x units from one end. Find the distance from this end to the center of mass.



total mass:

$$m = \int_0^7 (1+x^3)\,dx$$

$$= \left(x + \frac{x^4}{4}\right)\Big|_0^7$$

$$= \left(7 + \frac{7^4}{4}\right) - 0 = \frac{2429}{4}$$

moment:

$$M = \int_0^7 x(1+x^3)\,dx$$

$$= \int_0^7 (x+x^4)\,dx = \left(\frac{x^2}{2} + \frac{x^5}{5}\right)\Big|_0^7$$

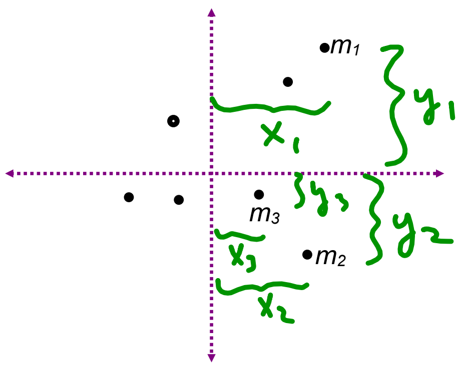
$$= \left(\frac{49}{2} + \frac{7^5}{5}\right) - 0 = \frac{33859}{10}$$

$$\Rightarrow \bar{x} = \frac{M}{m} = \frac{\frac{33859}{10}}{\frac{2429}{4}} = \frac{33859}{10} \cdot \frac{4}{2429}$$

$$= \frac{135436}{24290} \approx \boxed{5.576 \text{ units}}$$

Consider a discrete set of 2-d masses.

How do we find the center of mass (the geometric center) $(\overline{x},\overline{y})$?



$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

where $M_y = \sum_{i=1}^{n} x_i m_i$

$M_x = \sum_{i=1}^{n} y_i m_i$ } moments

total mass

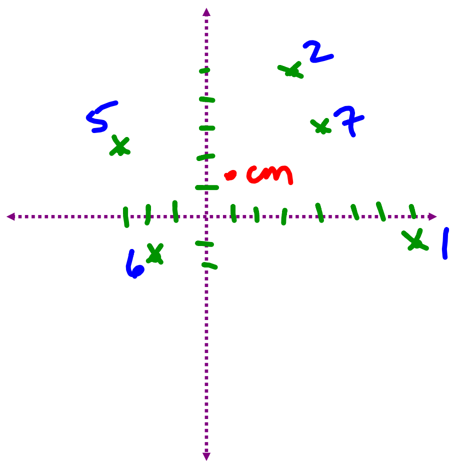
$$m = \sum_{i=1}^{n} m_i$$

note: M_y is the "x moment" measured from the y-axis

EX 3

The masses and coordinates of a system of particles are given by the following: 5, (-3,2); 6, (-2,-2); 2, (3,5); 7, (4,3); 1, (7,-1). Find the moments of this system with respect to the coordinate axes and find the center of mass.

(mass is written in blue)



$$M_y = \sum_{i=1} x_i m_i$$

$$= 5(-3) + 2(3) + 7(4) + 6(-2) + 1(7)$$

$$M_x = 5(2) + 2(5) + 7(3) + 6(-2) + 1(-1)$$

$$\Rightarrow M_y = -15 + 6 + 28 - 12 + 7$$

$$= -27 + 6 + 7 + 28$$

$$= 14$$

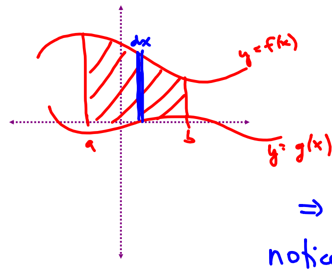
$$M_x = 10 + 10 + 21 - 12 - 1 = 41 - 13 = 28$$

total mass $= m = 5 + 2 + 7 + 6 + 1 = 21$

$$\bar{x} = \frac{M_y}{m} = \frac{14}{21} = \boxed{\frac{2}{3}} \qquad \bar{y} = \frac{M_x}{m} = \frac{28}{21} = \boxed{\frac{4}{3}}$$

Now, consider a continuous 2-d region (a lamina) that has constant (homogeneous) density everywhere. How do we find the center of mass $(\bar{x},\bar{y})$?

It's still true $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$



total mass
$$\boxed{m = \int_a^b \delta\,(f(x)-g(x))\,dx}$$
(density · area)

δ = density (mass per area unit)

$\Rightarrow \delta \cdot \text{area} = \text{mass}$

notice: $\int_a^b (f(x)-g(x))\,dx$ = total area

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} x_i m_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i \left(f(x_i)-g(x_i)\right)\Delta x_i\,(\delta)$$
$$= \delta \int_a^b x\,(f(x)-g(x))\,dx$$

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{\cancel{\delta}\int_a^b x\,(f(x)-g(x))\,dx}{\cancel{\delta}\int_a^b (f(x)-g(x))\,dx}$$

$$\boxed{\bar{x} = \frac{\int_a^b x\,(f(x)-g(x))\,dx}{\int_a^b (f(x)-g(x))\,dx}}$$
x-coord. of center of mass for 2d lamina

note: not dependent on δ.
(geometry is the most important)

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} y_i m_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{f(x_i)+g(x_i)}{2}\right)}_{\text{avg y-value in that little bit of area/mass.}} \underbrace{\left(\delta\,(f(x_i)-g(x_i)\right)\Delta x_i}_{\text{mass}}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\delta}{2}\left(f^2(x_i)-g^2(x_i)\right)\Delta x_i$$

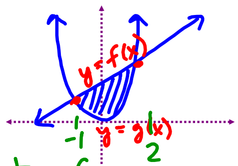
$$M_x = \int_a^b \frac{\delta}{2}\left(f^2(x)-g^2(x)\right)dx$$

$$\Rightarrow \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\cancel{\delta}\int_a^b \left(f^2(x)-g^2(x)\right)dx}{\cancel{\delta}\int_a^b (f(x)-g(x))\,dx}$$

$$\boxed{\bar{y} = \frac{\int_a^b \left(f^2(x)-g^2(x)\right)dx}{2\int_a^b (f(x)-g(x))\,dx}}$$
y-coord of center of mass for 2d lamina

EX 4

Find the centroid of the region bounded by $y = x^2$ and $y = x+2$.



$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x\,(f(x)-g(x))\,dx}{\int_a^b (f(x)-g(x))\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b (f^2(x)-g^2(x))\,dx}{\int_a^b (f(x)-g(x))\,dx}$$

pts of intersectn

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

$$m = \int_{-1}^{2} (x+2-x^2)\,dx$$
$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_1^2$$
$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$
$$= -\frac{9}{3} + 8 - \frac{1}{2} = 4\tfrac{1}{2} = \boxed{\frac{9}{2}}$$

$$M_y = \int_{-1}^{2} x(x+2-x^2)\,dx$$
$$= \int_{-1}^{2} (-x^3+x^2+2x)\,dx = \left(-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right)\Big|_1^2$$
$$= \left(\cancel{-4} + \frac{8}{3} + \cancel{4}\right) - \left(-\frac{1}{4} - \frac{1}{3} + 1\right)$$
$$= -1 + \frac{9}{3} + \frac{1}{4} = 2\tfrac{1}{4} = \boxed{\frac{9}{4}}$$

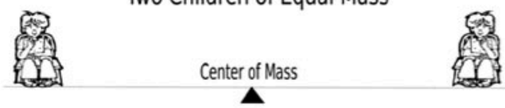
$$M_x = \frac{1}{2}\int_{-1}^{2} \left((x+2)^2 - (x^2)^2\right)dx$$
$$= \frac{1}{2}\int_{-1}^{2} (-x^4 + x^2 + 4x + 4)\,dx$$
$$= \frac{1}{2}\left(-\frac{x^5}{5} + \frac{x^3}{3} + 2x^2 + 4x\right)\Big|_{-1}^2$$
$$= \frac{1}{2}\left[\left(-\frac{32}{5} + \frac{8}{3} + 8 + 8\right) - \left(\frac{1}{5} - \frac{1}{3} + 2 - 4\right)\right]$$
$$= \frac{1}{2}\left[-\frac{33}{5} + \frac{9}{3} + 18\right]$$
$$= \frac{1}{2}\left[21 - \frac{33}{5}\right] = \frac{105-33}{10} = \boxed{\frac{72}{10}}$$

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{9/4}{9/2} = \frac{9}{4}\cdot\frac{2}{9} = \boxed{\frac{1}{2}}$$

$$\bar{y} = \frac{M_x}{m} = \frac{72/10}{9/2} = \frac{72}{10}\cdot\frac{2}{9} = \boxed{\frac{8}{5}}$$

centroid $\left(\frac{1}{2}, \frac{8}{5}\right)$

Two Children of Equal Mass



One Highschooler and One Elementary Schooler

One Baby and One Cow

One Planet and One Star