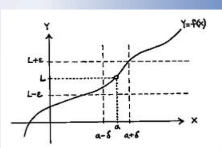
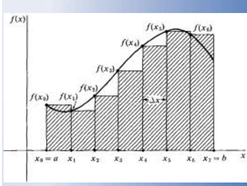


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

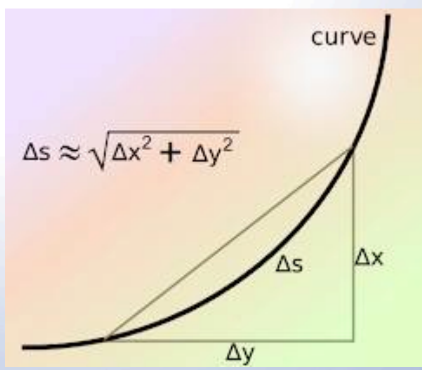
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

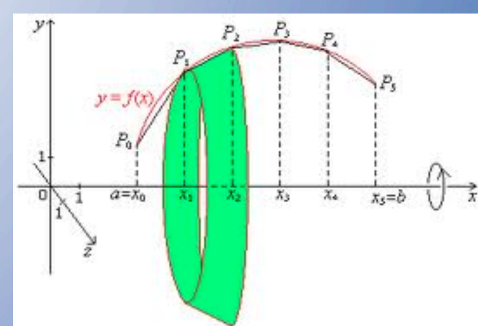


$$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Length of a Curve and Surface Area





Length of a Plane Curve

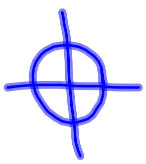
A plane curve is a curve that lies in a two-dimensional plane. We can define a plane curve using parametric equations. This means we define both x and y as functions of a parameter.

Parametric equations $\theta \in [0, 2\pi)$

ex $\boxed{y = \sin\theta,\ x = \cos\theta}$ ⇒ this is parametric eqns for a unit circle centered at origin in xy-plane

$\sin^2\theta + \cos^2\theta = 1$

parameter is θ



$\boxed{x^2 + y^2 = 1}$

Definition

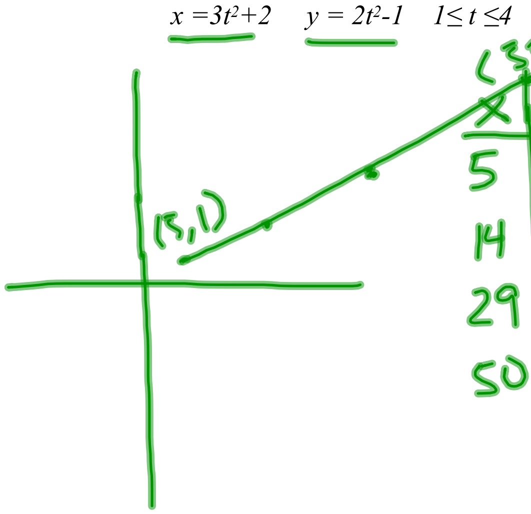
A plane curve is smooth if it is given by a pair of parametric equations $x = f(t)$, and $y = g(t)$, t is on the interval [a,b] where f' and g' exist and are continuous on [a,b] and $f'(t)$ and $g'(t)$ are not simultaneously zero on (a,b).

(smooth has something to do w/ differentiability)

31B Length Curve

EX 1 Sketch the graph of the curve given by these parametric equations.

$x = 3t^2+2 \quad y = 2t^2-1 \quad 1 \leq t \leq 4$



x	y	t
5	1	1
14	7	2
29	17	3
50	31	4

$x = 3t^2+2$

$x-2 = 3t^2$

$\frac{x-2}{3} = t^2$

$\Rightarrow$ substitute into "y" eqn

$y = 2\left(\frac{x-2}{3}\right)-1$

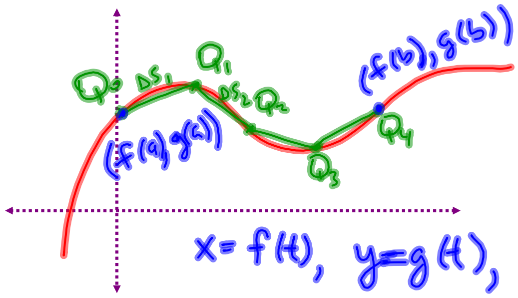
$y = \frac{2}{3}x - \frac{4}{3} - 1$

$$\boxed{\begin{array}{l} y = \frac{2}{3}x - \frac{7}{3} \\ x \in [5, 50] \end{array}}$$

Arc length

s = arc length

We can approximate the length of a plane curve by adding up lengths of linear segments between points on the curve.



$x = f(t)$, $y = g(t)$, $t \in [a,b]$

zoom in

Δs_i = little bit of arc length

① $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

use this in parametric form when $x = f(t)$, $y = g(t)$, $t \in [a,b]$

② $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

use this when $y = f(x)$ and $x \in [a,b]$

③ $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

use this when $x = h(y)$ and $y \in [c,d]$

$\Delta s_i \approx \Delta w_i$

$\Delta w_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

and $\Delta x_i = f(t_i) - f(t_{i-1})$

$\Delta y_i = g(t_i) - g(t_{i-1})$

① $f'(t_i) \approx \frac{f(t_i) - f(t_{i-1})}{\Delta t_i}$

② $g'(t_i) \approx \frac{g(t_i) - g(t_{i-1})}{\Delta t_i}$

$\Rightarrow \Delta x_i \approx f'(t_i)\Delta t_i$

$\Delta y_i \approx g'(t_i)\Delta t_i$

$\Rightarrow \Delta w_i \approx \sqrt{(f'(t_i)\Delta t_i)^2 + (g'(t_i)\Delta t_i)^2}$

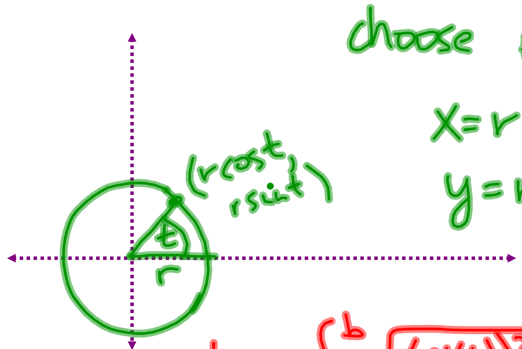
$\Delta w_i = \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\ \Delta t_i$

$\Rightarrow L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta w_i$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{(f'(t_i))^2 + (g'(t_i))^2}\ \Delta t_i\right)$

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt}$$

EX 2 Find the circumference of the circle $x^2 + y^2 = r^2$.



choose parametric eqns.

$x = r\cos t = f(t)$

$y = r\sin t = g(t)$

note: x^2+y^2
$= r^2\cos^2 t + r^2\sin^2 t$
$= r^2(\cos^2 t + \sin^2 t)$
$= r^2$

$\Rightarrow$ param. eqns are good

$t \in [0, 2\pi)$

$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$

$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt$

$f'(t) = -r\sin t$

$g'(t) = r\cos t$

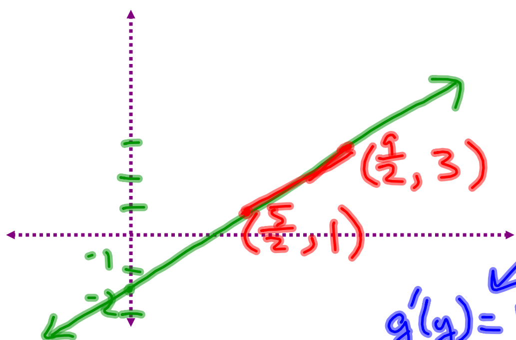
$= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$

$= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)}\,dt$

$= \int_0^{2\pi} r\,dt = r\int_0^{2\pi} dt$

$= r\left(t\Big|_0^{2\pi}\right) = r(2\pi) = \boxed{2\pi r}$

EX 3 Find the length of the line segment on $2y - 2x + 3 = 0$ between $y = 1$ and $y = 3$. Check your answer using the distance formula.



$2y = 2x - 3$

$y = x - \frac{3}{2}$

need to use arc length formula

$\boxed{x = y + \frac{3}{2}}$ $\quad g'(y) = 1 \quad x = g(y)$

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

where $x = g(y)$

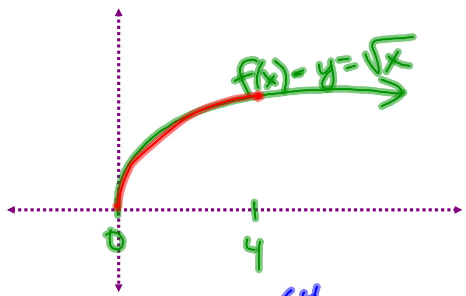
$$L = \int_1^3 \sqrt{1 + (1)^2}\, dy$$

$$= \sqrt{2}\int_1^3 dy = \sqrt{2}\left(y\Big|_1^3\right) = \sqrt{2}(3-1) = \boxed{2\sqrt{2}}$$

using distance formula:

$$L = \sqrt{\left(\frac{9}{2} - \frac{5}{2}\right)^2 + (3-1)^2} = \sqrt{4 + 4} = \sqrt{8} = \boxed{2\sqrt{2}}$$

EX 4 Find the arc length of the curve $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$.



use arc length that has dx

$$L = \int_0^4 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$L = \int_0^4 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx$$

$$\boxed{L = \int_0^4 \sqrt{\frac{4x+1}{4x}}\, dx}$$

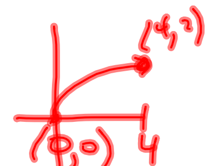
not necessarily "doable"

---

switch to dy integral

we have $y = \sqrt{x}$, from $x = 0$ to $x = 4$

this is the same $\to$ $x = y^2$, $(0, 0)$ $(4, 2)$



$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

$x = y^2$

$\frac{dx}{dy} = 2y$

$$= \int_0^2 \sqrt{1 + (2y)^2}\, dy$$

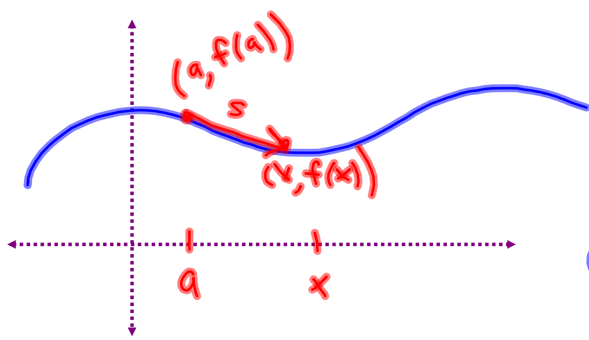
$$\boxed{L = \int_0^2 \sqrt{1 + 4y^2}\, dy}$$

not doable w/ u-sub. however we can do this integral w/ Calc 2 technique

<u>Surface Area</u>

<u>Differential of Arc Length</u>

Let f(x) be continuously differentiable on [a,b]. Start measuring arc length from (a,f(a)) up to (x,f(x)), where a is a real number. Then, the arc length is a function of x.



accumulate arc length

$$L = \underbrace{s(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt}$$

can take derivative wrt x:

$$\frac{ds}{dx} = \frac{d}{dx}\left(\int_a^x \sqrt{1+(f'(t))^2}\, dt \right)$$

$$\frac{ds}{dx} = \sqrt{1+(f'(x))^2}$$

① $\boxed{ds = \sqrt{1+(f'(x))^2}\, dx}$

a little bit of arc length is fn on right

if $y = f(x)$

also

② $ds = \sqrt{(f'(t))^2+(g'(t))^2}\, dt$

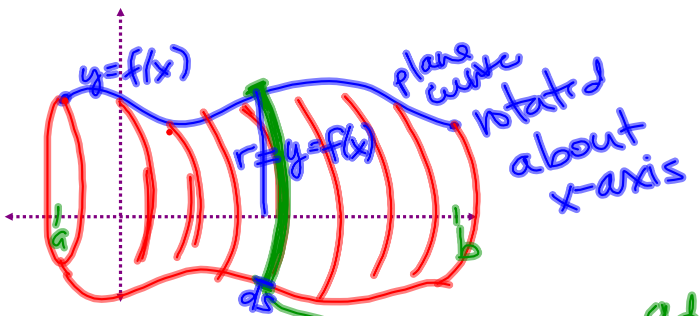
if $x = f(t),\ y = g(t)$

③ $ds = \sqrt{1+(h'(y))^2}\, dy$

if $x = h(y)$

Surface Area of a Surface of Revolution

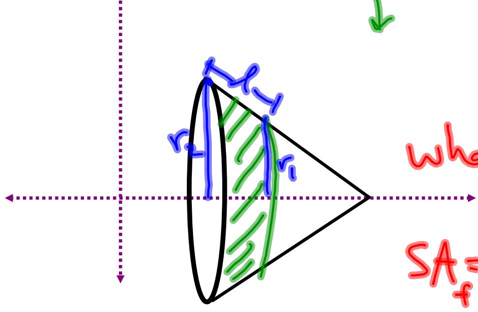
Rotate a plane curve about an axis to create a hollow three-dimensional solid.
Find the surface area of the solid.



think of green piece as a band of paper (or rubber band)

add up all such paper bands to get total surface area

Frustrum of a cone



what is surface area of that band? (frustrum)

$SA_f = 2\pi\,(\text{avg radius}) \cdot (\text{slant height})$

avg radius: $\frac{r_1+r_2}{2}$

slant height $= \ell$

$SA_f = 2\pi\left(\frac{r_1+r_2}{2}\right)\ell$

$$SA_{\substack{\text{whole}\\\text{surface}}} = \lim_{n\to\infty} \sum_{i=1}^{n} SA_f$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i \Delta s_i$$

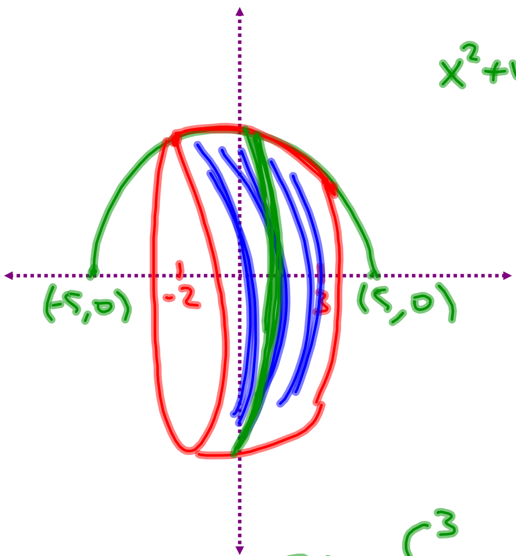
$$= \int_a^b 2\pi r \, ds$$

$$\boxed{SA = \int_a^b 2\pi f(x) \underbrace{\sqrt{1+(f'(x))^2}\, dx}_{ds}}$$

use when $y=f(x)$ and rotated about x-axis (makes $r=f(x)$)

EX 4 Find the area of the surface generated by revolving $y = \sqrt{25-x^2}$ on the interval [-2,3] about the x-axis.

half circle



$x^2+y^2=25$

notice: given $y=f(x)$

$$SA=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$f'(x)=\cancel{\tfrac{1}{2}}(25-x^2)^{-1/2}(-\cancel{2}x)$$

$$=\frac{-x}{\sqrt{25-x^2}}$$

$$SA=\int_{-2}^{3} 2\pi\left(\sqrt{25-x^2}\right)\sqrt{1+\left(\frac{-x}{\sqrt{25-x^2}}\right)^2}\,dx$$

$$=2\pi\int_{-2}^{3}\sqrt{(25-x^2)\left(1+\frac{x^2}{25-x^2}\right)}\,dx$$

$$=2\pi\int_{-2}^{3}\sqrt{25-\cancel{x^2}+\cancel{x^2}}\,dx$$

$$=2\pi\int_{-2}^{3}\sqrt{25}\,dx=10\pi\int_{-2}^{3}dx$$

$$=10\pi\left(x\Big|_{-2}^{3}\right)$$

$$=10\pi\left(3-(-2)\right)$$

$$=\boxed{50\pi}\text{ units}^2$$

EX 5 Find the area of the surface generated by revolving $x = 1-t^2$, $y = 2t$, on the t-interval $[0,1]$ about the x-axis.

$\underbrace{x = 1-t^2}_{f(t)}$, $\underbrace{y = 2t}_{g(t)}$

if $y=f(x)$ and rotate about x-axis

$$SA = \int_a^b 2\pi \underbrace{f(x)}_{r} \underbrace{\sqrt{1+(f'(x))^2}\,dx}_{ds}$$

other form of SA (if $y=g(t)$, $x=f(t)$) $t \in [a,b]$

$$SA = \int_a^b 2\pi\, g(t)\sqrt{(g'(t))^2+(f'(t))^2}\,dt$$

$$SA = \int_0^1 2\pi(2t)\sqrt{(2)^2+(-2t)^2}\,dt$$

$$= 4\pi\int_0^1 t\sqrt{4t^2+4}\,dt$$

$$= 4\pi\int_0^1 t\sqrt{4(t^2+1)}\,dt$$

$$= 4\pi(2)\int_0^1 t\sqrt{t^2+1}\,dt$$

$u = t^2+1$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$t=0,\ u=0^2+1=1$

$t=1,\ u=1^2+1=2$

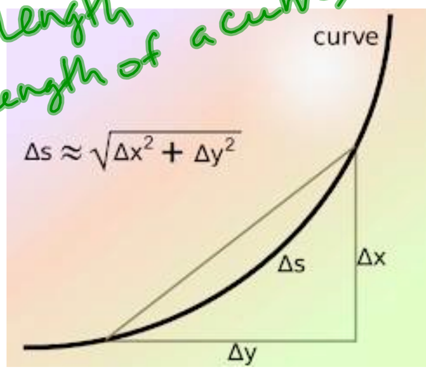
$$= 8\pi\int_1^2 \sqrt{u}\left(\tfrac{1}{2}\right)du$$

$$= 4\pi\int_1^2 u^{1/2}\,du$$

$$= 4\pi\left(\tfrac{2}{3}u^{3/2}\right)\Big|_1^2 = \frac{8\pi}{3}\left(2^{3/2}-1^{3/2}\right)$$

$$= \boxed{\frac{8\pi}{3}(2\sqrt{2}-1)}\ \text{units}^2$$

arc
Length
(Length of a curve)



Surface area
of frustrum