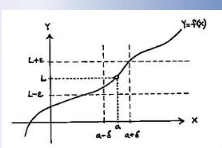
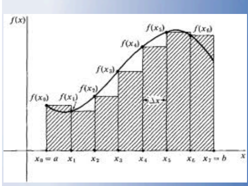


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

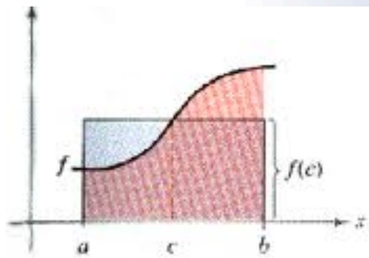
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



$$\lim_{max\,\Delta x_i \to 0} \sum_{1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

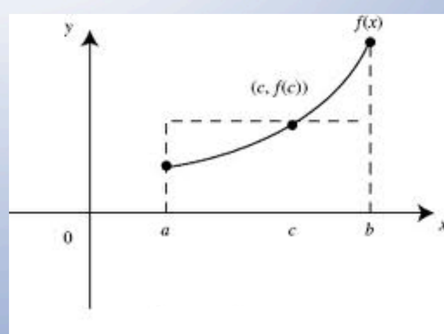
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Mean Value Theorem for Integrals



Mean Value Rectangle:

$$f(c)(b - a) = \int_a^b f(x)\,dx$$

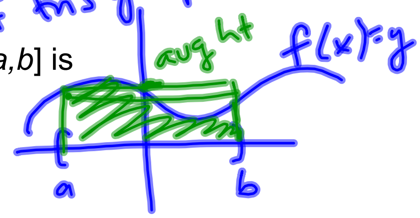


Definition Average Value of a Function

(avg height of fn's graph)

If f is integrable on $[a,b]$, then the average value of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$



EX 1   Find the average value of this function on [0,3] $f(x)=\frac{x}{\sqrt{x^2+16}}$

$a=0,\ b=3$

$$\text{avg ht} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$$

$$= \frac{1}{3}\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$$

u-sub:

$u = x^2+16$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$x=0,\ u=0^2+16=16$

$x=3,\ u=3^2+16=25$

$$= \frac{1}{3}\left(\frac{1}{2}\right)\int_{16}^{25}\frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{6}\int_{16}^{25} u^{-1/2}\,du$$

$$= \frac{1}{6}\left(2u^{1/2}\right)\Big|_{16}^{25}$$

$$= \frac{1}{3}\left(\sqrt{25}-\sqrt{16}\right)$$

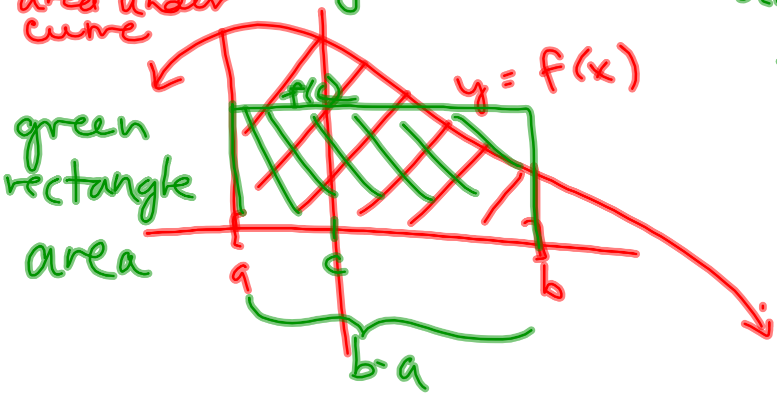
$$= \frac{1}{3}(5-4) = \boxed{1/3}$$

avg ht of fn is $\frac{1}{3}$

Mean Value Theorem for Integrals

If f is continuous on $[a,b]$ there exists a value c on the interval (a,b) such that

$$\underbrace{\int_a^b f(t)\,dt}_{\text{area under curve}} = \underbrace{f(c)(b-a)}_{\text{green rect. area}} .$$



avg ht on $[a,b]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,dt$

$f(c) =$ avg ht.

EX 2 Find the values of c that satisfy the MVT for integrals on $[0,1]$.

$a=0,\ b=1$

$f(x) = x(1-x)$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(c) = \int_0^1 x(1-x)\,dx = \int_0^1 x - x^2\,dx$$

$$c(1-c) = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$c - c^2 = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right)$$

$$c - c^2 = \frac{1}{6} \Longleftrightarrow c^2 - c + \frac{1}{6} = 0$$

$$c = \frac{1 \pm \sqrt{1 - 4\left(\frac{1}{6}\right)}}{2}$$

$$c = \frac{1 \pm \sqrt{1/3}}{2}$$

$$c = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$c = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$$

$$\boxed{c = \frac{1}{2} \pm \frac{\sqrt{3}}{6}}$$

EX 3 Find values of c that satisfy the MVT for integrals on $[3\pi/4, \pi]$.

$a = \frac{3\pi}{4},\ b = \pi$

$f(x) = \cos(2x - \pi)$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$(b-a)f(c) = \int_a^b f(x)\,dx$$

$$\frac{\pi}{4}\cos(2c-\pi) = \int_{3\pi/4}^{\pi} \cos(2x-\pi)\,dx$$

aside

$$\int_{3\pi/4}^{\pi} \cos(2x-\pi)\,dx$$

$u = 2x - \pi$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$x = 3\pi/4,\ u = 2\left(\frac{3\pi}{4}\right) - \pi = \pi/2$

$x = \pi,\ u = 2\pi - \pi = \pi$

$$= \int_{\pi/2}^{\pi} \cos u \left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\left(\sin u \Big|_{\pi/2}^{\pi}\right)$$

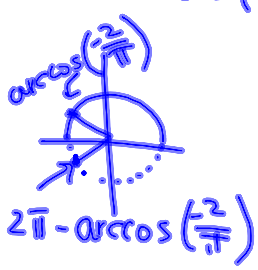
$$= \frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{2}\right)$$

$$= \frac{1}{2}(0 - 1) = -1/2$$

$$\frac{\pi}{4}\cos(2c-\pi) = -\frac{1}{2}$$

$$\cos(2c-\pi) = -\frac{1}{2}\left(\frac{4}{\pi}\right)$$

$$\cos(2c-\pi) = \frac{-2}{\pi}$$



$$2c - \pi = \arccos\left(\frac{-2}{\pi}\right),\quad 2\pi - \arccos\left(\frac{-2}{\pi}\right)$$

$$2c = \pi + \arccos\left(\frac{-2}{\pi}\right),\quad 3\pi - \arccos\left(\frac{-2}{\pi}\right)$$

$$c = \frac{\pi}{2} + \frac{1}{2}\arccos\left(\frac{-2}{\pi}\right),\quad \frac{3\pi}{2} - \frac{1}{2}\arccos\left(\frac{-2}{\pi}\right)$$

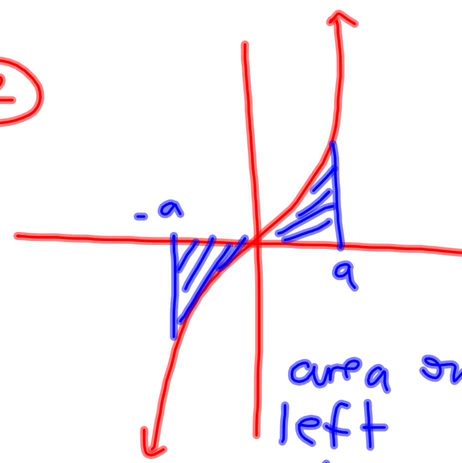
Is c in $[3\pi/4, \pi]$?

Symmetry Theorem

1. If f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

2. If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.

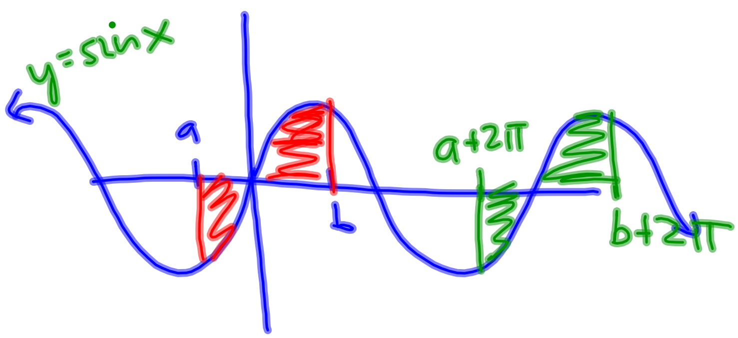
area from -a to a = 2(area from 0 to a)



area on left = −(area of right side)

Theorem

If f is a periodic function with period p, then $\int_{a+p}^{b+p} f(x)\,dx = \int_{a}^{b} f(x)\,dx$.



EX 4 $\displaystyle\int_{-\pi/2}^{\pi/2} \underbrace{x^2\sin^2\left(x^3\right)\cos\left(x^3\right)}_{\text{even}} dx$

x^3 odd

$\sin^2(x^3)$ even

$\cos(x^3)$ even

$\left(\cos((-x)^3) = \cos(-x^3) = \cos(x^3)\right)$

x^2 even

$$= 2\int_0^{\pi/2} x^2 \sin^2(x^3)\cos(x^3)\,dx$$

$u = \sin(x^3)$

$\frac{du}{dx} = \cos(x^3)(3x^2)$

$\frac{1}{3}du = x^2\cos(x^3)\,dx$

$x = 0,\ u = \sin(0^3) = 0$

$x = \pi/2,\ u = \sin\left(\left(\pi/2\right)^3\right)$

$$= 2\int_0^{\sin(\pi^3/8)} u^2 \left(\tfrac{1}{3}\right)du$$

$$= \frac{2}{3}\left(\frac{u^3}{3}\right)\Bigg|_0^{\sin(\pi^3/8)}$$

$$= \frac{2}{9}\left(\sin^3(\pi^3/8) - 0\right)$$

$$= \boxed{\frac{2}{9}\sin^3\left(\pi^3/8\right)}$$

EX 5 $\displaystyle\int_{-\pi/2}^{\pi/2} \underbrace{\underbrace{x}_{\text{odd}}\underbrace{\sin^2\left(x^3\right)}_{\text{even}}\underbrace{\cos\left(x^3\right)}_{\text{even}}}_{\text{odd fn}} dx$

$= 0$

28B MVT Integrals