Solving Equations Numerically

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

\[ \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x_k = \int_a^b f(x) \, dx \]

\[ \int_a^b F(x) \, dx = F(b) - F(a) \]
Three numeric methods for solving an equation numerically:

① Bisection Method
② Newton's Method
③ Fixed-point Method
① Bisection Method Algorithm

Let \( f(x) \) be a continuous function and let \( a_l \) and \( b_r \) be numbers satisfying \( a_l < b_r \), and \( f(a_l) \cdot f(b_r) < 0 \).

Let \( E \) denote the desired bound for the error \( |r - m| \).

Repeat steps 1 to 5 for \( n = 1, 2, \ldots \) until \( h_n < E \)

1. Calculate \( m_n = \frac{a_m + b_m}{2} \)

2. Calculate \( f(m_n) \) and if \( f(m_n) = 0 \) then STOP.

3. Calculate \( h_n = \frac{|b_n - a_n|}{2} \) (for error testing).

4. If \( f(a_n) \cdot f(m_n) < 0 \), then set \( a_{n+1} = a_n \) and \( b_{n+1} = m_n \).

5. If \( f(a_n) \cdot f(m_n) > 0 \), then set \( a_{n+1} = m_n \) and \( b_{n+1} = b_n \).

Pros: always works

Cons: converges slowly
EX 1: Approximate the real root to 2 decimal places. \( f(x) = x^4 + 5x^3 + 1 \) on \([-1,0]\)

(use Bisection Method)

\[
\begin{array}{cccccc}
\text{count} & a_n & b_n & m_n & f(a_n) & f(b_n) & f(m_n) \\
1 & -1 & 0 & -0.5 & -3 & 1 & 0.4375 \\
2 & -1 & -0.5 & -0.75 & -3 & 0.4375 & -0.7929688 \\
3 & -0.75 & -0.5 & -0.625 & -0.7929688 & 0.4375 & -0.0681152 \\
4 & -0.625 & -0.5 & -0.5625 & -0.0681152 & 0.4375 & 0.21022034 \\
5 & -0.625 & -0.5625 & -0.59375 & -0.0681152 & 0.21022034 & 0.07768345 \\
6 & -0.625 & -0.59375 & -0.609375 & -0.0681152 & 0.07768345 & 0.00647169 \\
\end{array}
\]

\( \approx 0 \) (up to 2 decimal places)
Newton's Method Algorithm

Let \( f(x) \) be a differentiable function and let \( x_1 \) be an initial approximation to the root, \( r \) of \( f(x) = 0 \). Let \( E \) denote a bound for the error \( |r-x_n| \).

Repeat the following step for \( n = 1, 2, \ldots \) until \( |x_{n+1} - x_n| < E \)

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

Pros:
- fast
- elegant

Cons:
- it doesn’t always converge (dependent on initial guess)
EX 2 Use Newton's method to approximate a root of \( 7x^3 + 2x - 5 = 0 \) to 5 decimal places.

\[
f(x) = 7x^3 + 2x - 5 \\
f'(x) = 21x^2 + 2
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{7x_n^3 + 2x_n - 5}{21x_n^2 + 2}
\]

\[
x_{n+1} = \frac{14x_n^3 + 5}{21x_n^2 + 2}
\]

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<th>( x_n )</th>
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\[ \Rightarrow \text{answer is } \approx 0.78792 \]
Warning on Newton's Method:

Original guess is $x_0$, looking to find $r$. 

Diagram showing iterations $x_0$, $x_1$, $x_2$, $x_3$.