Mean Value Theorem for Derivatives

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

\[ \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x = \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
20B Mean Value Theorem

Mean Value Theorem for Derivatives
If \( f \) is continuous on \( [a,b] \) and differentiable on \( (a,b) \), then there exists at least one \( c \) on \( (a,b) \) such that

\[
\frac{f(b) - f(a)}{b-a} = f'(c)
\]

EX 1 Find the number \( c \) guaranteed by the MVT for derivatives for
\( g(x) = (x+1)^3 \) on \([-1,1]\)

\( a = -1, b = 1 \)
\( g(b) = (1+1)^3 = 8, g(-1) = 0 \)

Secant line slope:
\[
\frac{8-0}{1-(-1)} = \frac{8}{2} = 4
\]

Tangent line slope:
\( g'(x) = 3(x+1)^2 \)

\[
4 = 3(c+1)^2
\]
\[
\frac{4}{3} = (c+1)^2
\]
\[
\pm \frac{2}{\sqrt{3}} = (c+1)
\]
\[
c = -1 \pm \frac{2}{\sqrt{3}} = -\frac{\sqrt{3} \pm 2}{\sqrt{3}}
\]

(need \( c \) value in \([-1,1]\))

\[
-\frac{\sqrt{3} + 2}{\sqrt{3}} \approx 0.3 \quad c \in [-1,1]
\]
\[
-\frac{\sqrt{3} - 2}{\sqrt{3}} \approx -2.7 \quad \not\in [-1,1] \quad \Rightarrow c = -\frac{\sqrt{3} + 2}{\sqrt{3}}
\]
EX 2  For \( g(x) = \frac{x-4}{x-3} \), decide if we can use the MVT for derivatives on \([0,5]\) or \([4,6]\). If so, find \( c \). If not, explain why.

(1) \([0,5]\) contains the VA at \( x=3 \)

\( g(x) \) not continuous on \([0,5]\)

\( \Rightarrow \) MVT for deriv. does not apply!

(2) on \([4,6]\), \( g(x) \) continuous & differentiable.

\( \Rightarrow \) can use MVT for deriv.

\( g(x) = \frac{x-4}{x-3} \)  \( a=4, b=6 \), \( g(c) = \frac{2}{3} \), \( g(4) = 0 \)

Slope of secant line: \( \frac{\frac{2}{3} - 0}{6-4} = \frac{\frac{2}{3}}{2} = \frac{1}{3} \)

Slope of tangent \( g'(x) = \frac{(x-3)(1)-(x-4)(1)}{(x-3)^2} = \frac{-3+4}{(x-3)^2} = \frac{1}{(x-3)^2} \)

\( \frac{1}{(c-3)^2} = \frac{1}{3} \)

\( (c-3)^2 = 3 \)

\( c-3 = \pm \sqrt{3} \)

\( c = 3 + \sqrt{3} \)

\( \Rightarrow c \in [4,6] \)

\( \boxed{c = 3 + \sqrt{3}} \)
EX 3  For \( f(x) = \csc x \) on \([-\pi/2, \pi/2]\), use the MVT for derivatives to find \( c \).

\[ f(x) = \frac{1}{\sin x} \]

notice: \( \sin 0 = 0 \)

\( \Rightarrow f(x) \) has discontinuity (VA) at \( x = 0 \)

\( x = 0 \in [-\pi/2, \pi/2] \Rightarrow \) we cannot apply MVT for deriv.

pretend we didn't check conditions

\[ f'(x) = -\csc x \cot x \]

secant line: \( a = -\pi/2, \ b = \pi/2 \)

slope \( f(-\pi/2) = -1, \ f(\pi/2) = 1 \)

\( \frac{1-(-1)}{\pi/2 - (-\pi/2)} = \frac{2}{\pi} \)

\[ -\csc x \cot x = \frac{2}{\pi} \]

\[ -\frac{\cos x}{\sin^2 x} = \frac{2}{\pi} \]

\[ -\cos x = \frac{2}{\pi} \sin^2 x \]

\[ -\cos x = \frac{2}{\pi} (1-\cos^2 x) \]

\( \frac{2}{\pi} \cos^2 x - \cos x - \frac{2}{\pi} = 0 \)

use quadratic formula to solve for \( \cos x \).

get an actual answer for \( x \).
Theorem B

If \( f'(x) = g'(x) \) for all \( x \) on the interval \((a,b)\),
then there exists a real number, \( c \), such that \( f(x) = g(x) + c \)
for all \( x \) in the interval \((a,b)\).
20B Mean Value Theorem

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

picture for mVT of derivatives