Local Extrema

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

\[ \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x = \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
### Definition

Let $S$ be the domain of $f$ such that $c$ is an element of $S$.

Then,

1) $f(c)$ is a **local maximum** value of $f$ if there exists an interval $(a,b)$ containing $c$ such that $f(c)$ is the maximum value of $f$ on $(a,b) \cap S$.

2) $f(c)$ is a **local minimum** value of $f$ if there exists an interval $(a,b)$ containing $c$ such that $f(c)$ is the minimum value of $f$ on $(a,b) \cap S$.

3) $f(c)$ is a **local extreme value** of $f$ if it is either a local maximum or local minimum value.
How do we find the local extrema?

**First Derivative Test**

Let $f$ be continuous on an open interval $(a, b)$ that contains a critical $x$-value.

1. If $f'(x) > 0$ for all $x$ on $(a, c)$ and $f'(x) < 0$ for all $x$ on $(c, b)$, then $f(c)$ is a local maximum value.
2. If $f'(x) < 0$ for all $x$ on $(a, c)$ and $f'(x) > 0$ for all $x$ on $(c, b)$, then $f(c)$ is a local maximum value.
3. If $f'(x)$ has the same sign on both sides of $c$, then $f(c)$ not a maximum nor a minimum value.
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EX 1  Determine local maximum and minimum points for \( y = 2x^2 - 5x + 3 \).

\[ y' = \frac{4x - 5}{x} = 0 \]
\[ x = \frac{5}{4} \]

\( y' \) changes from negative to positive at \( x = \frac{5}{4} \), hence a local minimum.

\[ y \left( \frac{5}{4} \right) = 2 \left( \frac{5}{4} \right)^2 - 5 \left( \frac{5}{4} \right) + 3 \]
\[ = \frac{25}{8} - \frac{25}{4} + 3 \]
\[ = -\frac{25}{8} + 3 = \frac{1}{8} \]

Global minimum at \( \left( \frac{5}{4}, \frac{1}{8} \right) \).

EX 2  Find all local maximum and minimum points for \( f(x) = \frac{1}{2} x + \sin x \) on \([0, 2\pi]\).

\[ f'(x) = \frac{1}{2} + \cos x = 0 \]
\[ \cos x = -\frac{1}{2} \]
\[ x = \frac{2\pi}{3}, \frac{4\pi}{3} \]

\( f''(x) \) changes from positive to negative at \( x = \frac{2\pi}{3} \), hence a local maximum.
\( f''(x) \) changes from negative to positive at \( x = \frac{4\pi}{3} \), hence a local minimum.

Test:
\[ x = \frac{\pi}{6}, \quad (+) \quad (+) \]
\[ x = \frac{\pi}{2}, \quad \frac{1}{2} - 1 \]
\[ x = \frac{3\pi}{2}, \quad \frac{1}{2} + 0 \]

\[ f \left( \frac{2\pi}{3} \right) = \frac{1}{2} \left( \frac{2\pi}{3} \right) + \sin \left( \frac{2\pi}{3} \right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \]

\[ f \left( \frac{4\pi}{3} \right) = \frac{1}{2} \left( \frac{4\pi}{3} \right) + \sin \left( \frac{4\pi}{3} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \]
**Theorem: Second Derivative Test**

Let $f'$ and $f''$ exist at every point on the interval $(a,b)$ containing $c$ and $f'(c) = 0$.

1) If $f''(c) < 0$, then $f(c)$ is a local maximum.
2) If $f''(c) > 0$, the $f(c)$ is a local minimum.

**EX 3** Find all critical points for $f(x) = x^3 - 3x^2 + 1$.

$$f'(x) = 3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0, 2$$

**Test:**
- $x = 1$, $-\left(+\right)$
- $x = 3$, $+\left(+\right)$

$$f''(x) = 6x - 6$$
$$f''(0) = -6 < 0 \implies \text{concave down at } x = 0 \implies \text{max}$$
$$f''(2) = 12 - 6 = 6 > 0 \implies \text{concave up at } x = 2 \implies \text{min}$$

**Critical Points**
- $(0, 1)$ local max
- $(2, -3)$ local min

$$f(x) = x^3 - 3x^2 + 1$$
- $f(0) = 1$
- $f(2) = 8 - 4(3) + 1 = -3$$
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EX 4 Find local and global extrema for \( y = x^2 + \frac{1}{x^2} \) on \([-2, 2]\).

Note: there’s a VA at \( x=0 \) (we expect all derivatives to also be undefined at \( x=0 \))

\[
\frac{dy}{dx} = 2x + \frac{-2}{x^3} = 0
\]

\[
2x^4 - 2 = 0
\]

\[
x^4 = 1
\]

\[
x = \pm 1
\]

\[
y'' = 2 + \frac{-2(-3)}{x^4}
\]

\[
y'' = \frac{2x^4 + 6}{x^4} > 0 \text{ always}
\]

\[
y = x^2 + \frac{1}{x^2}
\]

\[
y(\pm 1) = 1 + 1 = 2
\]

\[
y(\pm 2) = 4 + \frac{1}{4} = \frac{17}{4}
\]

\[
\Rightarrow \text{no global max}
\]

(because graph goes up to \( \infty \))

\[
\text{global min pts } (\pm 1, 2)
\]
EX 5  Let $f$ be continuous such that $f'$ has the following graph.

Try to sketch a graph of $f(x)$ and answer these questions.

a) Where is $f$ increasing?

b) Where is $f$ decreasing?

c) Where is $f$ concave up?

d) Where is $f$ concave down?

e) Where are inflection points?

f) Where are local max/min values?

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[Diagram showing local extrema with labels for stationary points and singular points.]