Monotonicity and Concavity

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i = \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} F(x) \, dx = F(b) - F(a) \]
Definition

Let $f$ be defined on an interval $I$, (open, closed or neither), we say that:

1) $f$ is **increasing** on $I$ if for every $x_1, x_2$ in $I$ $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
2) $f$ is **decreasing** on $I$ if for every $x_1, x_2$ in $I$ $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
3) $f$ is **strictly monotonic** on $I$ if it is either increasing or decreasing on $I$.

**Monotonicity Theorem**

Let $f$ be continuous on the interval, $I$ and differentiable everywhere inside $I$.

1) if $f'(x) > 0$ for all $x$ on the interval, then $f$ is increasing on that interval.
2) if $f'(x) < 0$ for all $x$ on the interval, then $f$ is decreasing on that interval.
EX 1 For each function, determine where $f$ is increasing and decreasing.

a) $f(x) = x^3 + 3x^2 - 12$ (cont everywhere)

$$f'(x) = 3x^2 + 6x = 0$$

$$3x(x + 2) = 0$$

$x = 0, x = -2$

$f$ increasing on $(-\infty, -2) \cup (0, \infty)$

$f$ decreasing on $(-2, 0)$

b) $f(x) = \frac{x-1}{x^2}$

(discontinuity at $x = 0$)

$$f'(x) = \frac{x^2(1) - (x-1)(2x)}{x^4}$$

$$= \frac{x^2 - 2x^2 + 2x}{x^4}$$

$$= \frac{-x^2 + 2x}{x^4} = \frac{x(-x+2)}{x^4} = 0$$

$$-x+2 = 0 \quad x = 2$$

(stationary pt $x$-value)

$f$ is increasing $(0, 2)$

$f$ is decreasing $(-\infty, 0) \cup (2, \infty)$
EX 2 Where is \( f(x) = \cos^2 x \) increasing and decreasing on the interval \([0,2\pi]\)?

\[
f'(x) = 2\cos x (-\sin x) = -2\cos x \sin x \quad (no \ singular \ pts)
\]

\[
-2\cos x \sin x = 0
\]

\[
-\sin(2x) = 0
\]

\[
\sin(2x) = 0
\]

\[
2x = 0, \pi, 2\pi, 3\pi, 4\pi
\]

\[
x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \quad (x-values \ for \ stationary \ pts)
\]

**sign line**

\[
\begin{array}{cccc}
0 & \frac{\pi}{2} & \pi & \frac{3\pi}{2} & 2\pi \\
- & + & - & + & \\
\end{array}
\]

\[
f'(x) \quad \text{test: } f'(x) = -2\sin x \cos x
\]

\[
\begin{array}{c}
1 \quad x = \frac{\pi}{4}, \quad -2(+)(-)
\\
2 \quad Q_2, \quad -2(+)(-)
\\
3 \quad Q_3, \quad -2(-)(-)
\\
4 \quad Q_4, \quad -2(-)(+)
\end{array}
\]

\(f\) is increasing on \((\frac{\pi}{2}, \pi) U (\frac{3\pi}{2}, 2\pi)\)

\(f\) is decreasing on \((0, \frac{\pi}{2}) U (\pi, \frac{3\pi}{2})\)
Definition

Let $f$ be differentiable on an open interval, $I$.

$f$ is **concave up** on $I$ if $f'(x)$ is increasing on $I$, and

$f$ is **concave down** on $I$ if $f'(x)$ is decreasing on $I$.

Concavity Theorem

Let $f$ be twice differentiable on an open interval, $I$.

If $f''(x) > 0$ for all $x$ on the interval, then $f$ is concave up on the interval.

If $f''(x) < 0$ for all $x$ on the interval, then $f$ is concave down on the interval.
EX 3 Determine where this function is increasing, decreasing, concave up and concave down.

\[ f(x) = 4x^3 - 3x^2 - 6x + 12 \]

\[ f'(x) = 12x^2 - 6x - 6 = 0 \quad \text{(no singular pts)} \]

\[ 6(2x^2 - x - 1) = 0 \]

\[ \text{Factored form of } f'(x): 6(2x+1)(x-1) = 0 \]

\[ 2x+1=0 \text{ or } x-1=0 \Rightarrow x = -\frac{1}{2}, 1 \]

**Sign of \( f'(x) \):**

\[ \begin{array}{c|c|c}
   \text{Line} & \frac{1}{2} & + \\
   \text{Inc.} & \text{Dec.} & \text{Inc.} \\
   \text{Test values:} & x = -1, & + (-)(-), \text{etc.} \\
   & x = 0, & + (+)(-), \text{etc.} \\
   & x = 2, & + (+)(+), \text{etc.} \\
\end{array} \]

\[ f''(x) = 24x - 6 = 0 \]

\[ 24x = 6 \Rightarrow x = \frac{1}{4} \]

**Concave down:** \( \frac{1}{4} \)

**Concave up:**

**Answer:**

- Increasing: \(( -\infty, -\frac{1}{2}) \cup (1, \infty)\)
- Decreasing: \(( -\frac{1}{2}, 1)\)
- Concave up: \(( \frac{1}{4}, \infty)\)
- Concave down: \(( -\infty, \frac{1}{4})\)
Inflection Point

Let $f$ be continuous at $c$. We call $(c, f(c))$ an inflection point of $f$ if $f$ is concave up on one side of $c$ and concave down on the other side of $c$.

Inflection points will occur at $x$-values for which $f''(x) = 0$ or $f''(x)$ is undefined.

Note: just because $f''(x) = 0$ or $f''(x)$ undefined does not mean that this is inflection pt.
EX 4 For this function, determine where it is increasing and decreasing,
where it is concave up and down, find all max/min and inflection points.

Use this information to sketch the graph.

\[ f(x) = 8x^{1/3} - x^{1/3} = 8\sqrt[3]{x} - x^{1/3} \]

\[ f'(x) = \frac{8}{3}x^{-2/3} - \frac{1}{3}x^{-1/3} \]

\[ x=0 \]

\[ f'(0) \text{ is undefined} \]

\[ f'(x) = \frac{8}{3}x^{-2/3} - \frac{1}{3}x^{-1/3} \]

\[ f''(x) = -\frac{16}{9}x^{-5/3} - \frac{1}{9}x^{-2/3} = -\frac{16}{9\sqrt[3]{x^2}} - \frac{4}{9\sqrt[3]{x^3}}(\frac{x}{x}) \]

Note: \( x \neq 0 \)

\[ \frac{-16-4x}{9x^{5/3}} = 0 \text{ when } x = -4 \]

Test:

\[ x = -8 \]

\[ \frac{-16+32}{9(-8)^{2/3}} = \frac{16}{9(-8)^{2/3}} > 0 \]

\[ x = 1 \]

\[ \frac{-16-4}{9} < 0 \]

\[ x = -4 \]

\[ \frac{-16+4}{9} = \frac{-12}{9} > 0 \]

Inflection pts:

\[ (-4, -1280), (0, 0) \]

Vertical inf. Pt.