Differentials and Approximations

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i = \int_a^b f(x) \, dx \]

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]
15.5B Differentials

Differentials and Approximations

We have seen the notation $\frac{dy}{dx}$ and we’ve never separated the symbols. Now, we’ll give meaning to $dy$ and $dx$ as separate entities.

We know \( \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \) gives the derivative (slope) of the function \( f(x) \) at \( x=x_0 \).

If \( \Delta x \) is really small, then \( \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \approx f'(x_0) \)

and \( f(x_0+\Delta x)-f(x) = f'(x_0)\Delta x \)

Differentials

Let \( y=f(x) \) be a differentiable function of \( x \). \( \Delta x \) is an arbitrary increment of \( x \).

\( dx = \Delta x \) (\( dx \) is called a differential of \( x \).)

\( \Delta y \) is actual change in \( y \) as \( x \) goes from \( x \) to \( x+\Delta x \).

\( \Delta y = f(x+\Delta x)-f(x) \)

i.e. \( \Delta y = f(x+\Delta x)-f(x) \) (\( \Delta y \) is the change in \( y \).)

\( dy = f'(x)dx \) (\( dy \) is called the differential of \( y \).)

\[
\frac{dy}{dx} = f'(x)
\]

\[
dy = f'(x) \, dx
\]
EX 1  Find $dy$.

a) $y = 4x^3 - 2x + 5$
\[ \frac{dy}{dx} = 12x^2 - 2 \implies dy = (12x^2 - 2)\,dx \]

b) $y = 2\sqrt{x^4 + 6x}$
\[ \frac{dy}{dx} = 2\left(\frac{1}{2}\right)(x^4 + 6x)^{-\frac{1}{2}}(4x^3 + 6) \]
\[ dy = \frac{(4x^3 + 6)}{\sqrt{x^4 + 6x}}\,dx \]

c) $y = \cos(x^3 - 5x + 11)$
\[ \frac{dy}{dx} = -\sin(x^3 - 5x + 11)(3x^2 - 5) \implies dy = \frac{-\sin(x^3 - 5x + 11)(3x^2 - 5)}{(3x^2 - 5)}\,dx \]

d) $y = (x^{10} + \sqrt{\sin(2x)})^2$
\[ \frac{dy}{dx} = 2(x^{10} + \sqrt{\sin(2x)}) \left(10x^9 + \frac{\cos(2x)(2x)}{2\sqrt{\sin(2x)}}\right) \]
\[ dy = \frac{2(x^{10} + \sqrt{\sin(2x)}) \left(10x^9 + \frac{\cos(2x)(2x)}{\sqrt{\sin(2x)}}\right)}{2\sqrt{\sin(2x)}}\,dx \]
Differentials can be used for approximations.

If \( f(\Delta x) = f(x+\Delta x) - f(x) \approx f'(x) \Delta x \),

then \( f(\Delta x) = f(x) + f'(x) \Delta x \).

**EX 2** Find a good approximation for \( \sqrt{9.2} \) without using a calculator.

Let \( f(x) = \sqrt{x} \), \( x = 9 \) \( \Delta x = \Delta x = 0.2 \)

\[
\begin{align*}
  f(x+\Delta x) & \approx f(x) + f'(x) \Delta x \\
  f(9.2) & \approx f(9) + f'(9)(0.2) \\
  & \approx \sqrt{9} + \frac{1}{2\sqrt{9}} (0.2) \\
  & = 3 + \frac{1}{6} (0.2) \\
  & = 3 + 0.0333 = 3.033 \\
  \Rightarrow \sqrt{9.2} & \approx 3.033
\end{align*}
\]
EX 3  Use differentials to approximate the increase in the surface area of a soap bubble when its radius increases from 4 inches to 4.1 inches.

\[
SA = f(r) = 4\pi r^2 \quad , \quad r = 4 \text{ in}, \quad \Delta r = dr = 0.1 \text{ in}
\]

Want to approximate \( \Delta f \).

\( \Delta f \approx df = f'(r)dr \quad , \quad f'(r) = 8\pi r \)

\[
\Delta f \approx 8\pi (4)(0.1) = 3.2\pi \approx 10.05 \text{ in}^2
\]

EX 4  The height of a cylinder is measured as 12 cm with a possible error of \( \pm 0.1 \) cm. Evaluate the volume of the cylinder with radius 4 cm and give an estimate for the possible error in this value.

\[
V = \pi r^2 h = 16\pi h \quad , \quad dh = \pm 0.1 \text{ cm}
\]

\[
\Delta V \approx dV = V'(h)dh = 16\pi (\pm 0.1)
\]

\[
dV = \pm 16\pi \approx 50.3 \text{ cm}^3
\]

\[
\Rightarrow V \approx 603.2 \pm 5.03 \text{ cm}^3
\]
if $\Delta x$ very small, $dy \approx dy$.

orig. orig. x-value + a little bit of change