Math 1090 ~ Business Algebra

Section 5.3 Future Value of Annuities

Objectives:
• Determine the future value of an ordinary annuity.
• Solve problems involving annuities.
An **annuity** is a financial plan characterized by regular payments.

### Ordinary Annuity
- Payments made at the end of each equal payment interval

### Annuity Due
- Payments made at the beginning of each equal payment interval

**Ex 1:** Suppose you invest $1000 at the end of each year for 5 years in an account that pays 10% interest compounded annually.

What is the value after 5 years?

- **end of year 1:** $1000

- **end of year 2:** $1000 \left(1 + 0.1\right) + 1000

- **end of year 3:** \(\frac{1000 \left(1+0.1\right)^2}{\text{from 1st yr deposit}} + 1000 \left(1+0.1\right) \text{ from 2nd yr deposit} + 1000 \text{ from 3rd yr deposit}\n
- **end of year 4:** \(1000 \left(1+0.1\right)^3 + 1000 \left(1+0.1\right)^2 + 1000 \left(1+0.1\right) + 1000\n
- **end of year 5:** \(1000 \left(1+0.1\right)^4 + 1000 \left(1+0.1\right)^3 + 1000 \left(1+0.1\right)^2 + 1000 \left(1+0.1\right) + 1000\n
Notice at end of 5th yr, we have 5 terms; and it's sum of geom. sequence

\(a_1 = 1000, \ d = (1+0.1), \ n = 5\)

We know that formula:

\[\text{Total Balance} = \frac{1000 \left(1 - (1 + 0.1)^5\right)}{1 - (1 + 0.1)}\]
Generally, for an ordinary annuity, the future value is

\[ S = \frac{R(1 - (1+r_c)^N)}{1 - (1+r_c)} \]

where \( r_c = \frac{r}{n} \)

\[ R = \text{monthly deposit} \]

\[ N = nt \]

\[ S = \frac{R(1 - (1+r_c)^N)}{-r_c} \]

\[ = \frac{R}{r_c} \left[ -1 + (1+r_c)^N \right] \]

\[ S = \frac{R[(1+r_c)^N - 1]}{r_c} \]

Future value of ordinary annuity (we make regular deposits/payments to an account; what is FV)

\[ S = \frac{a_1(1-a^n)}{1-a} \]

Sum of geometric sequence

Compound interest formula
Ex 2: A story of twins

a) At the end of college, Thelma invests $2000 at the end of each year for 8 years in an account that earns 10% compounded annually. After 8 years, she contributes nothing, but it continues to earn the same interest for 36 more years. How much does she have then?

\[
S = \frac{R((1+r_c)^N-1)}{r_c}
\]

1. **FV (8 yrs)**
   \[
   S = \frac{2000(1.1^8-1)}{0.1}
   \]
   \[
   S = \$22,871.78
   \]

2. **compound int.**
   \[
   S = 22,871.78(1+0.1)^{36} \approx \$707,027.91
   \]

\(S = \text{her total deposits:} \quad \$16,000\)

b) At the end of college Lewis invests nothing for 8 years. Then he puts $2000 into an account at the end of each year for 36 years earning 10% interest compounded annually. How much does he have then?

\[
S = \frac{R((1+r_c)^N-1)}{r_c}
\]

\[
S = 2000(1.1^{36}-1) \quad r_c = 0.1 \quad N = 1(36) = 36
\]

\(S = \text{his total deposits:} \quad 2000(36) = \$72,000\)

\(n=1\)
Ex 3: How much should be invested quarterly (at the end of each quarter) at 12% interest compounded quarterly to pay off a debt of $30,000 in 6 years?

\[ t = 6 \quad r_c = \frac{0.12}{4} = 0.03 \]

\[ r = 0.12 \quad n = 4 \]

\[ S = 30,000 \]

\[ 30,000 = R \left( \frac{(1 + 0.03)^{24} - 1}{0.03} \right) \]

\[ \frac{30,000(0.03)}{(1.03^{24} - 1)} = R \]

\[ R = \$ 871.42 \]

(Total deposits: \( 871.42 \times 24 = \$ 20,914.08 \))

\[ S = \frac{R((1+r_c)^N - 1)}{r_c} \]

Sinking Fund

\[ R = S \left( \frac{r_c}{(1+r_c)^N - 1} \right) \]

The payment that needs to be invested every pay period to pay off debt of \( S \) at the end.
Ex 4: Find the future value of an account with $100 deposited at the beginning of each month for 5 years into an account that pays 8% compounded monthly.

\[ R = \$100 \]
\[ n = 12 \]
\[ t = 5 \]
\[ r = 0.08 \]
\[ r_c = \frac{0.08}{12} = 0.006666666 \]
\[ S = 100(1+0.006666666)^{60} - \frac{1.006666666^{60} - 1}{0.006666666} \]
\[ \approx 7396.67 \]

(Future value of Annuity Due)

(1 deposits made at beginning of periods)

(looks like FV ordinary annuity w/ one extra payment)

(annuity w/ one extra payment)

(FV annuity due)