Math 1060 ~ Trigonometry

4 The Six Trigonometric Functions

Learning Objectives

In this section you will:

- Determine the values of the six trigonometric functions from the coordinates of a point on the Unit Circle.
- Learn and apply the reciprocal and quotient identities.
- Learn and apply the Generalized Reference Angle Theorem.
- Find angles that satisfy trigonometric function equations.
The Trigonometric Functions

In addition to the sine and cosine functions, there are four more.

**Trigonometric Functions:** Suppose \( \theta \) is an angle plotted in standard position and \( P(x, y) \) is the point on the terminal side of \( \theta \) which lies on the Unit Circle. The circular functions are defined as follows.

- The **sine** of \( \theta \), denoted \( \sin(\theta) \), is defined by \( \sin(\theta) = y \).
- The **cosine** of \( \theta \), denoted \( \cos(\theta) \), is defined by \( \cos(\theta) = x \).
- The **tangent** of \( \theta \), denoted \( \tan(\theta) \), is defined by \( \tan(\theta) = \frac{y}{x} \), provided \( x \neq 0 \).
- The **cosecant** of \( \theta \), denoted \( \csc(\theta) \), is defined by \( \csc(\theta) = \frac{1}{y} \), provided \( y \neq 0 \).
- The **secant** of \( \theta \), denoted \( \sec(\theta) \), is defined by \( \sec(\theta) = \frac{1}{x} \), provided \( x \neq 0 \).
- The **cotangent** of \( \theta \), denoted \( \cot(\theta) \), is defined by \( \cot(\theta) = \frac{x}{y} \), provided \( y \neq 0 \).

Ex 1: Assume \( \theta \) is \( \frac{\pi}{3} \) in this picture.

Find the six trigonometric functions of \( \theta \).

\[
\begin{align*}
\cos \theta &= \frac{1}{2} \\
\sin \theta &= \frac{\sqrt{3}}{2} \\
\tan \theta &= \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \\
\sec \theta &= \frac{1}{\frac{1}{2}} = 2 \\
\csc \theta &= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \\
\cot \theta &= \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}
\end{align*}
\]
Ex 2: Determine the tangent values for the first quadrant and each of the quadrant angles on this Unit Circle.

**NOTE:**

\[ \tan\left(\frac{\pi}{2}\right) \notin \mathbb{R} \]

\[ \tan\left(\frac{3\pi}{2}\right) \text{ is undefined!} \]

\[ \tan\pi = \frac{0}{0} = 0 \]
Reciprocal and Quotient Identities

**Reciprocal and Quotient Identities:**

- \( \tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} \), provided \( \cos(\theta) \neq 0 \); if \( \cos(\theta) = 0 \) then \( \tan(\theta) \) is undefined.
- \( \cot(\theta) = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)} \), provided \( \sin(\theta) \neq 0 \); if \( \sin(\theta) = 0 \) then \( \cot(\theta) \) is undefined.
- \( \sec(\theta) = \frac{1}{x} = \frac{1}{\cos(\theta)} \), provided \( \cos(\theta) \neq 0 \); if \( \cos(\theta) = 0 \) then \( \sec(\theta) \) is undefined.
- \( \csc(\theta) = \frac{1}{y} = \frac{1}{\sin(\theta)} \), provided \( \sin(\theta) \neq 0 \); if \( \sin(\theta) = 0 \) then \( \csc(\theta) \) is undefined.

Ex 3: Find the indicated value, if it exists.

a) \( \sec 30^\circ = \frac{2}{\sqrt{3}} \)

b) \( \csc \left( \frac{11\pi}{6} \right) = -\frac{1}{\frac{1}{2}} = -2 \)

c) \( \cot (2) = \frac{\cos 2}{\sin 2} \approx -0.458 \)

d) \( \tan \theta \), where \( \theta \) is any angle coterminal with 270°.

e) \( \cos \theta \), where \( \csc \theta = -2 \) and \( \frac{3\pi}{2} < \theta < 2\pi \).

f) \( \sin \theta \), where \( \tan \theta = 1 \) and \( \theta \) is in Q III.
Generalized Reference Angle Theorem

The values of the trigonometric functions of an angle, if they exist, are the same, up to a sign, as the corresponding trigonometric functions of the reference angle.

More specifically, if \( \alpha \) is the reference angle for \( \theta \), then \( \cos \theta = \pm \cos \alpha \), \( \sin \theta = \pm \sin \alpha \). The sign, + or −, is determined by the quadrant in which the terminal side of \( \theta \) lies.

Ex 4: Determine the reference angle for each of these. Then state the cosine and sine and tangent of each.

a) \(-225^\circ\)
\[ \alpha = 45^\circ \]
\[ \sin(-225^\circ) = \frac{\sqrt{2}}{2} \]
\[ \cos(-225^\circ) = -\frac{\sqrt{2}}{2} \]
\[ \tan(-225^\circ) = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1 \]

b) \(\frac{11\pi}{6}\)
\[ \alpha = \frac{\pi}{3} \]
\[ \cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{2}}{2} \]
\[ \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2} \]
\[ \tan\left(\frac{11\pi}{6}\right) = -\frac{1}{\sqrt{3}} \]

c) \(-\frac{3\pi}{4}\)
\[ \alpha = \frac{\pi}{4} \]
\[ \cos\left(-\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \]
\[ \sin\left(-\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \]
\[ \tan\left(-\frac{3\pi}{4}\right) = 1 \]
Finding Angles that Satisfy Cosine and Sine Equations

Ex 5: Find all of the angles on the unit circle which satisfy the given equation.

\[ \theta \in [0, 2\pi) \]

a) \( \sin \theta = 0 \)
\[ \theta = 0, \pi \]

b) \( \cos \theta = -\frac{1}{2} \)
\[ \theta = 2\pi \frac{1}{3} \text{ or } \pi - \frac{1}{3} \]

or \( \theta = Q_2 \text{ or } Q_3 \)

c) \( \sin \theta = \frac{\sqrt{2}}{2} \)
\[ \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \]

Finding Angles that Satisfy Other Trigonometric Equations

Ex 6: Find all of the angles on the unit circle which satisfy the given equation.

a) \( \tan \theta = -1 \)
\[ \theta = \frac{3\pi}{4}, \frac{3\pi}{2} \]

b) \( \sec \theta = 2 \)
\[ \cos \theta = \frac{1}{2} \]
\[ \theta = \pi \frac{1}{3} \text{ or } \frac{2\pi}{3} \]

or \( \theta = Q_1 \text{ or } Q_4 \)

c) \( \cot \theta = 0 \)
\[ \implies \frac{\cos \theta}{\sin \theta} = \pm 1 \]
\[ \implies \cos \theta = 0 \]
\[ \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \]