Math 1060 ~ Trigonometry

28 Conic Sections: Hyperbolas

Learning Objectives

In this section you will:

• Define a hyperbola in a plane.
• Determine whether an equation represents a hyperbola or some other conic section.
• Graph a hyperbola from a given equation.
• Determine the center, vertices, foci and eccentricity of a hyperbola.
• Find the equation of a hyperbola from a graph or from stated properties.
Ex 1: Given the points $F_1(-5,0)$ and $F_2(5,0)$, plot several points such that the difference of the distances from $F_1$ and $F_2$ to each point is 4. Draw the curve connecting the points.

$$
\begin{align*}
7-3 &= 4 \\
8-4 &= 4 \\
9-5 &= 4 \\
12-8 &= 4
\end{align*}
$$

$$
e = \frac{c}{a} = \frac{\sqrt{5}}{2}
$$
Hyperbolas

General form: $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A$ and $B$ have opposite signs.

Given: two points (foci) and a distance ($c$).

Definition: A hyperbola is the set of all points in a plane such that for each point on the hyperbola, the difference of its distances from two fixed points is constant.

Vocabulary

Transverse axis

Asymptotes

Center

Foci
Standard Form of an Equation of a Hyperbola with Center at (0,0)

The variables $a$, $b$ and $c$ have a special relationship.

$$b^2 = c^2 - a^2 \quad \text{OR} \quad a^2 + b^2 = c^2$$
Ex 2: Write the equation of these hyperbolas in standard form.

a) \[
\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1
\]

b) \[
\frac{y^2}{3^2} - \frac{x^2}{5^2} = 1
\]

Ex 3: Determine the value of \( c \) for each hyperbola above and plot the foci.

**remember** \( a^2 + b^2 = c^2 \)

(a) \( 4 + 9 = c^2 \)
\[
c = \sqrt{13} \approx 3.6
\]
\[
e = \frac{\sqrt{13}}{2} \approx 1.80
\]

(b) \( 9 + 25 = c^2 \)
\[
c^2 = 34
\]
\[
c = \sqrt{34} \approx 5.8
\]
\[
e = \frac{\sqrt{34}}{3} \approx 1.94
\]
Translations of a Hyperbola

Standard Hyperbola  Translated Hyperbola

Center at (0,0)  Center at (h,k)

horiz. \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

vert. \[ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \]

\[ \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \]

Ex 4: Sketch each of these curves and locate the foci.

a) \( 4x^2 - 9y^2 = 36 \)

\[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \]

Center \((0,0)\)  \(a = 3\)  \(b = 2\)

\[ c^2 = a^2 + b^2 = 9 + 4 = 13 \]

\[ c = \sqrt{13} \approx 3.6 \]

\[ e = \frac{\sqrt{13}}{3} \approx 1.20 \]

b) \( 16(y+2)^2 - 9(x-3)^2 = 36 \)

\[ \frac{(y+2)^2}{9/4} - \frac{(x-3)^2}{16} = 1 \]

Center \((3,-2)\)  \(a = \frac{3}{2}\)  \(b = 2\)

\[ c^2 = \frac{9}{4} + 4 = \frac{25}{4} \]

\[ c = \frac{5}{2} \]

\[ e = \frac{\frac{5}{2}}{3/2} = \frac{5}{3} \approx 1.67 \]
Ex 5: Write an equation and sketch each of these.

a) The hyperbola such that the center is (-2,3), one of the asymptotes passes through (1,4) and it is vertically oriented.

\[
\frac{(y-3)^2}{1} - \frac{(x+2)^2}{9} = 1
\]

b) A hyperbola with vertices at (-4,3) and (2,3) and foci at (-6,3) and (4,3)

hyperbola is horizontal
Center (-1, 3)
\[ a = 3, \quad c = 5 \]
\[ a^2 + b^2 = c^2 \]
\[ 9 + b^2 = 25 \]
\[ b = 4 \]

\[
\frac{(x+1)^2}{9} - \frac{(y-3)^2}{16} = 1
\]

c = 1 + 9 \Rightarrow c = \sqrt{10}

\[ e = \frac{c}{a} = 3.16 \]
Ex 6: Write this equation in standard form, sketch it, including the foci.

\[ x^2 - 9y^2 - 4x - 18y - 14 = 0 \]

\[
\begin{align*}
(x-2)^2 - 9(y+1)^2 &= 9 \\
\frac{(x-2)^2}{9} - (y+1)^2 &= 1
\end{align*}
\]

Eccentricity of a Hyperbola

\[ e = c/a \]

For hyperbolas, \( c > a \)
Ex 7: Identify each of these equations as one of these:

C - Circle \((x^2 \text{ and } y^2 \text{ have exact same coefs.})\)

E - Ellipse that is not a circle (longer in which direction)

H - Hyperbola (facing which way) \((x^2 \text{ and } y^2 \text{ with opp. coefs.})\)

P - Parabola (facing which way) \((x^2 \text{ or } y^2, \text{ but not both squared})\)

i) \(9x^2 - 4y^2 - 36x + 8y - 4 = 0\)
   \(\cancel{\text{hyperbola}}\)

ii) \(y^2 + 4x - 2y - 11 = 0\)
   \(\underline{\text{parabola}}\)

iii) \(16x^2 + 16y^2 + 64x - 32y - 176 = 0\)
   \(\underline{\text{circle}}\)

iv) \(-9x^2 + 25y^2 - 54x - 50y - 281 = 0\)
   \(\underline{\text{hyperbola}}\)

v) \(9x^2 + 4y^2 - 18x + 16y - 11 = 0\)
   \(\underline{\text{ellipses}}\)

vi) \(x^2 - 6x + 8y - 7 = 0\)
   \(\underline{\text{parabola}}\)

vii) \(2x^2 + 3y^2 + 12x + 24y + 60 = 0\)
   \(\underline{\text{ellipses}}\)