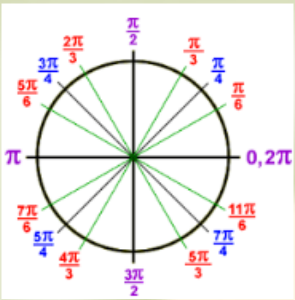
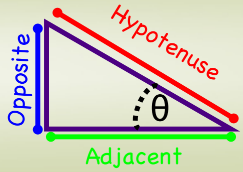
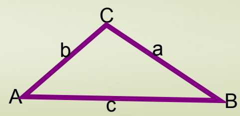
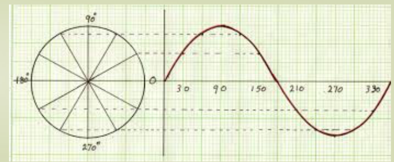








$\boxed{\sin^2 u + \cos^2 u = 1}$

$\boxed{\sin 2u = 2\sin u \cos u}$

$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$

$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$

Math 1060 ~ Trigonometry

20 Complex Products, Powers, Quotients and Roots

Learning Objectives

In this section you will:

- **Find the product, power, quotient and roots of complex numbers.**
- **Learn and apply DeMoivre's Theorem.**

Operations on Complex Numbers in Trigonometric (Polar) Form

To multiply two complex numbers, multiply the moduli and add the arguments.

To divide two complex numbers, divide the moduli and subtract the arguments.

Note: If the new argument is out of range, you will need to find a coterminal angle that is in the interval $[0, 2\pi)$.

1. $z_1 z_2 = r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$ $= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$

2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$ $= \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2)$

Ex 1: Let $z_1 = 3\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$ and $z_2 = 12\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

a) $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$

$= 3(12)\operatorname{cis}\left(\frac{5\pi}{3}+\frac{\pi}{6}\right)$

$\boxed{z_1 z_2 = 36\operatorname{cis}\left(\frac{11\pi}{6}\right)}$

b) $\dfrac{z_2}{z_1} = \dfrac{r_2}{r_1}\operatorname{cis}(\theta_2-\theta_1)$

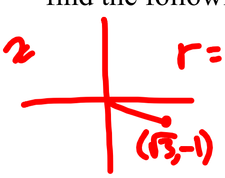
$= \frac{12}{3}\operatorname{cis}\left(\frac{\pi}{6}-\frac{5\pi}{3}\right)$

$= 4\operatorname{cis}\left(\frac{-9\pi}{6}\right) = 4\operatorname{cis}\left(\frac{-3\pi}{2}\right)$

$\frac{z_2}{z_1} = 4\operatorname{cis}\left(\frac{\pi}{2}\right)$

Ex 2: Let $z = \sqrt{3} - i$ and $w = -1 + i$. Convert these numbers to polar form and find the following, leaving the answers in polar form.

z

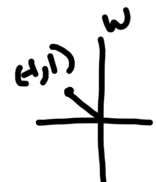


$r = \sqrt{(\sqrt{3})^2+(-1)^2}$
$= \sqrt{4} = 2$
$\tan\theta = \frac{-1}{\sqrt{3}}$

$\theta = \frac{11\pi}{6}$

$r = \sqrt{(-1)^2+1^2}$
$= \sqrt{2}$
$\tan\theta = \frac{1}{-1} = -1$

w



$\theta = \frac{3\pi}{4}$

a) zw b) $\dfrac{z}{w}$ c) w^2

(a)
$zw = 2(\sqrt{2})\operatorname{cis}\left(\frac{11\pi}{6}+\frac{3\pi}{4}\right)$

$zw = 2\sqrt{2}\operatorname{cis}\left(\frac{22\pi+9\pi}{12}\right)$

$= 2\sqrt{2}\operatorname{cis}\left(\frac{31\pi}{12}\right)$

$= 2\sqrt{2}\operatorname{cis}\left(\frac{31\pi}{12}-\frac{24\pi}{12}\right)$

$zw = 2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

(b)
$\frac{z}{w} = \frac{2}{\sqrt{2}}\operatorname{cis}\left(\frac{11\pi}{6}-\frac{3\pi}{4}\right)$

$= \sqrt{2}\operatorname{cis}\left(\frac{22\pi-9\pi}{12}\right)$

$= \sqrt{2}\operatorname{cis}\left(\frac{13\pi}{12}\right)$

(c)
$w^2 = w \cdot w$

$= \sqrt{2}\cdot\sqrt{2}\operatorname{cis}\left(2\cdot\frac{3\pi}{4}\right)$

$= 2\operatorname{cis}\left(\frac{3\pi}{2}\right)$

$w^2 = r^2\operatorname{cis}(2\theta)$

Powers of Complex Numbers in Trigonometric (Polar) Form

This is called **DeMoivre's Theorem**.

$$\boxed{z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)} = r^n \text{ cis}(n\theta)$$

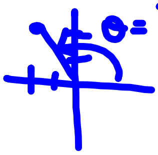
Ex 3: Use DeMoivre's Theorem to find these.

a) If $z = 3\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$, determine the value of z^4. (Answer in polar form.)

$$z^4 = 3^4\left(\cos\left(4\left(\frac{5\pi}{3}\right)\right) + i\sin\left(\frac{20\pi}{3}\right)\right) = 81 \text{ cis}\left(\frac{20\pi}{3}\right)$$

$$= \boxed{81 \text{ cis}\left(\frac{2\pi}{3}\right)}$$

b) If $w = -2+2i$, find w^5. (Answer in rectangular form.)



$\theta = \frac{3\pi}{4}$

$$r = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$w = 2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)$$

$$w^5 = (2\sqrt{2})^5 \text{ cis}\left(5\left(\frac{3\pi}{4}\right)\right) = 32(4)\sqrt{2} \text{ cis}\left(\frac{15\pi}{4}\right)$$

$$w^5 = 128\sqrt{2} \text{ cis}\left(\frac{7\pi}{4}\right)$$

$$w^5 = 128\sqrt{2}\cos\left(\frac{7\pi}{4}\right) + 128\sqrt{2}\sin\left(\frac{7\pi}{4}\right)i = 128\cancel{\sqrt{2}}\left(\frac{\cancel{\sqrt{2}}}{\cancel{2}}\right) + 128\cancel{\sqrt{2}}\left(\frac{-\cancel{\sqrt{2}}}{\cancel{2}}\right)i$$

$$w^5 = 128 - 128i$$

c) If $z = 2\text{cis}(230°)$, find z^8. (Answer in polar form with degrees.)

$$z^8 = 2^8 \text{ cis}(8(230°))$$

$$= 256 \text{ cis}(1840°)$$

$$z^8 = 256 \text{ cis}(40°)$$

$1840° - 720°$

$= 1120°$

$1120° - 720° = 400°$

$400° - 360° = 40°$

Determining n^{th} Roots of a Number

Since a root is simply a fractional exponent, we can use DeMoivre's Theorem to find the first root of a complex number in polar form. Each number will have n n^{th} roots. Find the rest of the roots by successively adding $\frac{2\pi}{n}$ or $\frac{360°}{n}$ until the n roots are found (keeping the angles on $[0,2\pi)$).

$$\boxed{w_k = \sqrt[n]{r}\,cis\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right),\ for\ k=0,1,\ldots,n-1}$$

Ex 4: Use DeMoivre's Theorem to find each of these.

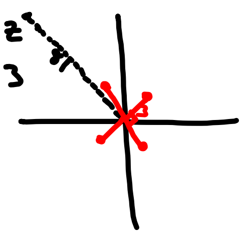
a) Find the four fourth roots of $z = 81\ \text{cis}\left(\frac{3\pi}{4}\right)$. (Answer in polar form.)

$n=4$

$z = 81\ cis\left(\frac{3\pi}{4}\right)$

$\sqrt[4]{z} = \sqrt[4]{81}\ cis\left(\frac{\frac{3\pi}{4}}{4} + \frac{2\pi}{4}k\right)$ $k=0,1,2,3$

$= 3\ cis\left(\frac{3\pi}{16} + \frac{\pi}{2}k\right),\ k=0,1,2,3$

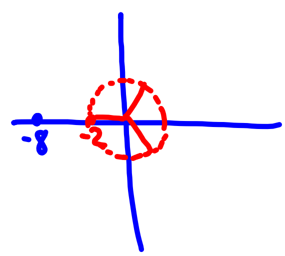


$\sqrt[4]{z} = 3cis\left(\frac{3\pi}{16}\right),\ 3cis\left(\frac{11\pi}{16}\right),$
$3cis\left(\frac{19\pi}{16}\right),\ 3cis\left(\frac{27\pi}{16}\right)$

b) Determine the three cube roots of -8. (Answer in rectangular form.)

$z=-8$

$\sqrt[3]{z} = \sqrt[3]{-8} = -2$



$\sqrt[3]{z} = 2cis(\pi),\ 2cis\left(\frac{\pi}{3}\right),$
$2cis\left(\frac{5\pi}{3}\right)$

$= -2,\ 2\left(\cos\frac{\pi}{3} + i\sin\left(\frac{\pi}{3}\right)\right),\ 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

$\sqrt[3]{z} = -2,\ 1+i\sqrt{3},\ 1-i\sqrt{3}$

check w/ DeMoivre's:

$z = -8 = 8\ cis\ \pi$

$\sqrt[3]{z} = \sqrt[3]{8}\ cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$ $k=0,1,2$

$= 2\ cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$ $k=0,1,2$

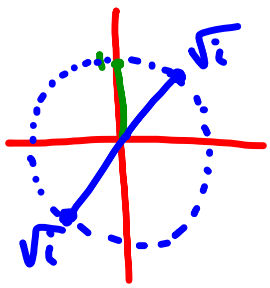
$= 2cis\left(\frac{\pi}{3}\right),\ 2cis(\pi),\ 2cis\left(\frac{5\pi}{3}\right)$ which matches what we got above

At last we are able to answer a question from our first lesson in complex numbers.

Ex 5: Find the two square roots of *i*.

$(n=2)$ $\quad r=1$

$\theta = \pi/2$



$$z = i = 1\,\text{cis}\left(\frac{\pi}{2}\right)$$

$$\sqrt{z} = \sqrt{1}\,\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{2}k\right) \quad k=0,1$$

$$= \text{cis}\left(\frac{\pi}{4} + \pi k\right) \quad k=0,1$$

$$= \text{cis}\left(\frac{\pi}{4}\right),\ \text{cis}\left(\frac{5\pi}{4}\right)$$

$$\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

sanity check:

$$\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$$

$$= \frac{2}{4} + \underbrace{i\left(\frac{2}{4}\right) + i\left(\frac{2}{4}\right)} + i^2\left(\frac{2}{4}\right)$$

$$= \frac{1}{2} + i + -1\left(\frac{1}{2}\right) = i \quad \checkmark$$