Math 1060 ~ Trigonometry

18 Graphing Polar Equations

Learning Objectives

In this section you will:

- Learn techniques for graphing polar equations.
- Graph polar equations.
What do these equations represent?

1. $\theta = \beta$ \hspace{1cm} (\(\beta\) constant)
   \hspace{1cm} (radial line through the pole/origin)

2. $r \cos \theta = a$ \hspace{1cm} $x = a$
   \hspace{1cm} (a constant)
   \hspace{1cm} vertical line

3. $r \sin \theta = b$ \hspace{1cm} $y = b$
   \hspace{1cm} (b constant)
   \hspace{1cm} horizontal line

These are all lines!
What about these?

1. $r = 2a \cos \theta$ (a constant)  
   $r^2 = 2a(r \cos \theta) \iff x^2 + y^2 = 2ax$

2. $r = 2b \sin \theta$ (b constant)  
   Circle centered at $(0, b)$ w/ radius $b$

3. $r = 2a \cos \theta + 2b \sin \theta$ (a,b constants)  
   Circle centered at $(a, b)$ w/ radius of $\sqrt{a^2 + b^2}$

Ex 1:

$r = 4 \cos \theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
<th>$2\pi/3$</th>
<th>$3\pi/4$</th>
<th>$\pi$</th>
<th>$5\pi/4$</th>
<th>$3\pi/2$</th>
<th>$7\pi/4$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>4</td>
<td>$2\sqrt{2}$</td>
<td>2</td>
<td>0</td>
<td>$-2\sqrt{2}$</td>
<td>$-4$</td>
<td>$-2\sqrt{2}$</td>
<td>0</td>
<td>$2\sqrt{2}$</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

$y(\frac{\pi}{2})$

$(4, 0) = (-4, \pi)$

$(2\sqrt{2}, \frac{3\pi}{4}) = (-2\sqrt{2}, \frac{\pi}{4})$

$(0, \frac{\pi}{2}) = (0, \frac{3\pi}{2})$

$(\frac{3\pi}{4}, 2\sqrt{2}) = (\frac{5\pi}{4}, -2\sqrt{2})$
Symmetry

Symmetry with respect to the line \( \theta = \pi/2 \)

Replace \((r, \theta)\) with \((r, \pi-\theta)\) or \((-r, -\theta)\):
If an equivalent equation results, the graph has this type of symmetry.

Symmetry with respect to the polar axis (\(\theta = 0\)):

Replace \((r, \theta)\) with \((r, -\theta)\) or \((-r, \pi-\theta)\):
If an equivalent equation results, the graph has this type of symmetry.

Symmetry with respect to the pole (origin):

Replace \((r, \theta)\) with \((-r, 0)\) or \((r, \pi+\theta)\):
If an equivalent equation results, the graph has this type of symmetry.

If a polar equation passes a symmetry test, then its graph definitely exhibits that symmetry.
However, if a polar equation fails a symmetry test, then its graph may or may not have that kind of symmetry.
Zeros and maximum $r$-values

Other helpful tools in graphing polar equations are knowing the values for $\theta$ for which $|r|$ is maximum and those for which $r = 0$.

Ex 2: Graph \[ r = \frac{1}{2} + \cos \theta \]

Symmetry:
1. Replace $(r, \theta)$ with $(-r, -\theta)$:
   \[ r = \frac{1}{2} + \cos(-\theta) \]
   \[ r = \frac{1}{2} + \cos(\theta) \]
   $\Rightarrow$ \[ r = \frac{1}{2} + \cos \theta \]

2. Replace $(r, \theta)$ with $(r, \theta)$:
   \[ r = \frac{1}{2} + \cos(\theta) \]
   symmetry wrt $\theta = 0$
   (x-axis)
   $\Rightarrow$ \[ r = \frac{1}{2} + \cos \theta \]

3. Replace $(r, \theta)$ with $(-r, \theta)$:
   \[ -r = \frac{1}{2} + \cos \theta \]

$|r|$ maximum:
\[ r = \frac{1}{2} + \cos \theta \]

Zero of $r$:
\[ \theta = \frac{1}{2} + \cos \theta \]
\[ \cos \theta = \frac{1}{2} \]
\[ \theta = \frac{\pi}{3}, \frac{5\pi}{3} \]

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
<th>$2\pi/3$</th>
<th>$3\pi/4$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{-\sqrt{3}}{2}$</td>
</tr>
</tbody>
</table>

Limaçon
Limaçons

\[ r = a \pm b \cos \theta \]
\[ r = a \pm b \sin \theta \]

\( a, b \) constants
\( a > 0, \ b > 0 \)
(all these picture examples are the cosine varieties)

\[ \frac{a}{b} \geq 2 \]
Convex
limaçon

\[ 1 < \frac{a}{b} < 2 \]
Dimpled
limaçon

\[ \frac{a}{b} = 1 \]
Cardioid
-always passes through pole

\[ \frac{a}{b} < 1 \]
Limaçon
with inner loop
Ex 3: Graph \( r = 3 \sin 2\theta \)

Symmetry: by replacing \((r, \theta) \rightarrow (-r, -\theta)\)

\[-r = 3 \sin(-2\theta)\]
\[-r = -3 \sin(2\theta)\]
\[r = 3 \sin(2\theta) \checkmark\]

Symmetry wrt \(\theta = \frac{\pi}{2}\) (y-axis)

\(r\) | maximum:
\(r\) is max when \(\sin(2\theta) = \pm 1\) \(\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}\)
\(\theta = \frac{\pi}{4}, \frac{3\pi}{4}\)
\((3, \frac{\pi}{4})\), \((-3, \frac{3\pi}{4})\)

Zero of \(r\):
\(0 = 3 \sin(2\theta)\)
\(\sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi\)
\(\theta = 0, \frac{\pi}{2}\)

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(0)</th>
<th>(\frac{\pi}{8})</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
<td>0</td>
<td>(\frac{3\sqrt{2}}{2})</td>
<td>(\frac{3\sqrt{3}}{2})</td>
<td>3</td>
<td>(\frac{3\sqrt{2}}{2})</td>
<td>0</td>
</tr>
</tbody>
</table>

\(r = 3 \sin(2\theta)\)

\(\frac{3\sqrt{2}}{2} \approx 2.12\)
\(\frac{3\sqrt{3}}{2} \approx 2.6\)
\((-3, \frac{3\pi}{4})\)
Roses

\[
\begin{align*}
  r &= a \sin (n \theta), \\
  r &= a \cos (n \theta).
\end{align*}
\]

or

\[a \text{ constant} \quad n \text{ constant} \]

If \( n \) is odd, the rose is \( n \)-petalled. If \( n \) is even, the rose is \( 2n \)-petalled.

\[n = 2 \quad n = 3 \quad n = 4 \quad n = 5\]

No reason to limit ourselves to \( n \) integer:

\[n = 1/2 \quad n = 1/3 \quad n = 2/3 \quad n = 1/4\]

\[n = 3/4 \quad n = 1/5 \quad n = 2/5 \quad n = 3/5\]

Or even rational:

\[n = e \quad n = \pi \quad n = \sqrt{2}\]