Math 1060 ~ Trigonometry

13 Solving Trigonometric Equations

Learning Objectives

In this section you will:

- Use inverse trigonometric functions to solve right triangles.
- Use inverse trigonometric functions to solve for angles in trigonometric equations.
- Write complete real solutions to equations containing a single trigonometric function.
- Evaluate exact solutions in the interval $[0, 2\pi)$.
- Use inverse trigonometric functions to solve real-world applications.
The inverse functions allow us to calculate angles in a right triangle, given two of the sides.

Ex 1: Determine the acute angles in a 3-4-5 right triangle.

\[
\sin \theta = \frac{3}{5} \quad \text{or} \quad \theta = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87^\circ
\]

\[
\tan \beta = \frac{4}{3} \Rightarrow \beta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13^\circ
\]

Ex 2: If a 50-meter rope is attached to the top of a 20-meter pole for a tight-rope event, what angle does the rope make with the ground?

\[
\sin \alpha = \frac{20}{50} \Rightarrow \alpha = \sin^{-1}\left(\frac{2}{5}\right) \approx 23.58^\circ
\]
We can also solve trigonometric equations for angles in radians.

**Remember:** $x = \sin^{-1}(a)$ returns a single, principal value and $\sin x = a$ will have an infinite number of solutions, if defined.

Sample: Solve for $x$.

1. $x = \sin^{-1}\left(-\frac{1}{2}\right)$ (has only one answer)
   
   
   \[
   x = -\frac{\pi}{6}
   \]

2. $\sin x = -\frac{1}{2}$ (has infinite solutions)
   
   \[
   x = \begin{cases} 
   -\frac{\pi}{6} + 2n\pi, & n \in \mathbb{Z} \\
   \frac{5\pi}{6} + 2n\pi, & n \in \mathbb{Z}
   \end{cases}
   \]

Ex 3: Solve these for $x$, where $x$ is in radians. State the solution on the interval $[0, 2\pi)$ and then state the general solution for all angles which provide a solution to the equation.

a) $\sqrt{2}\sin x - 1 = 0$

\[
\sqrt{2}\sin x = 1 \\
\sin x = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{1}\right) \\
\sin x = \frac{\sqrt{2}}{2}
\]

1. $x = \frac{\pi}{4}, \frac{3\pi}{4}$
   
   \[
   x \in \left[0, 2\pi\right)
   \]

2. $x = \left\{\frac{\pi}{4} + 2n\pi \right\}$

b) $\sec^2 x = 4$

\[
\frac{1}{\cos^2 x} = 4 \\
\cos^2 x = \frac{1}{4} \\
\cos x = \pm \frac{1}{2}
\]

1. $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$
   
   \[
   x \in \left[0, 2\pi\right)
   \]

2. $x = \left\{\pm\frac{\pi}{3} + 2n\pi \right\}$

Note:

$\sec^2 x = (\sec x)^2$
Ex 4: State the general solution for each of these.

a) \( \tan^2 x - 3 = 1 \)
\[
\tan^2 x = 4 \\
\tan x = \pm 2
\]
\[
x = \tan^{-1} 2 \\
\tan^{-1}(-2) \\
\tan^{-1}(-2) + \pi \\
\tan^{-1}(-2) + \pi \text{ (in Q3)}
\]
\[
x = \begin{cases} 
\tan^{-1} 2 + n\pi & n \in \mathbb{Z} \\
\tan^{-1}(-2) + n\pi & n \in \mathbb{Z}
\end{cases}
\]

b) \( \cos(2x) = -\frac{\sqrt{3}}{2} \)
\[
2x = \begin{cases} 
\frac{5\pi}{6} + 2n\pi \\
\frac{7\pi}{6} + 2n\pi
\end{cases} \\
\text{where } n \in \mathbb{Z}
\]
\[
x = \begin{cases} 
\frac{5\pi}{12} + n\pi \\
\frac{7\pi}{12} + n\pi
\end{cases}
\]

Ex 5: State all radian values where the line \( y = 2 \) intersects with the function \( y = \sec x \).

when is \( 2 = \sec x \)?

\[
\cos x = \frac{1}{2} \\
x = \pm \frac{\pi}{3} + 2n\pi \\
\text{where } n \in \mathbb{Z}
\]