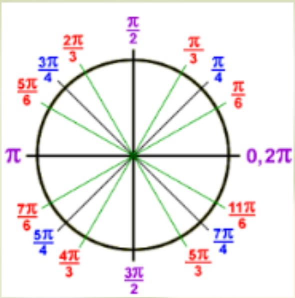
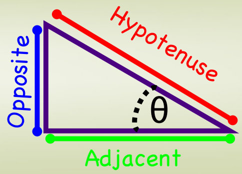
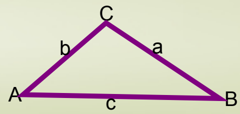
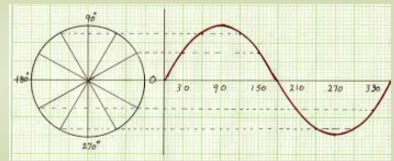








$\sin^2 u + \cos^2 u = 1$

$\sin 2u = 2\sin u \cos u$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

Math 1060 ~ Trigonometry

11 Multiple Angle Identities

Learning Objectives

In this section you will:

- **Learn the double and half angle identities for sine, cosine and tangent.**
- **Find trigonometric values of double and half angles.**
- **Verify identities involving double and half angles.**
- **Learn and apply the power reduction formulas for sine and cosine.**
- **Learn and apply product/sum formulas.**

The double angle identities are easy to generate using the identities for the sum of two angles.

sin (2θ) = sin ($\theta + \theta$) (use sum formula/ identity)

$= \sin\theta\cos\theta + \cos\theta\sin\theta$

$= 2\sin\theta\cos\theta$

$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

cos (2θ) = cos ($\theta + \theta$)

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

$\Rightarrow \boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$

Note:

because $\cos^2\theta = 1 - \sin^2\theta$

and $\sin^2\theta = 1 - \cos^2\theta$

we get 2 other varieties of double angle identity for cosine

$\cos(2\theta) = 1 - 2\sin^2\theta$

or $\cos(2\theta) = 2\cos^2\theta - 1$

tan (2θ) = tan ($\theta + \theta$)

$$= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

Double Angle Identities: For all applicable angles θ,

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

Why do we need the double angle identities? Do they allow us to compute exact values of any angles?

- Simplify expressions.
- Solve equations with $2x$.

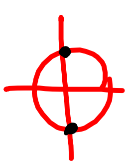
Ex 1: Solve this equation for values of x on the interval $[0, 2\pi)$.

$\sin(2x) + \cos(x) = 0$ (use double angle identity)

$2\sin x \cos x + \cos x = 0$

$\cos x \,(2\sin x + 1) = 0$

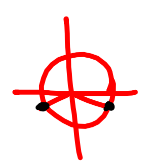
$\cos x = 0$ or $2\sin x + 1 = 0$



$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$



$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Starting with two forms of the double angle identity for the cosine, we can generate half-angle identities for the sine and cosine.

1

$\cos(2\theta) = 1 - 2\sin^2\theta$

$\cos(2\theta) - 1 = -2\sin^2\theta$

2

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) + 1 = 2\cos^2\theta$

Power Reduction Formulas: For all angles θ,

- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$
- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}}$$

half-angle identity

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

$$\boxed{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \pm\sqrt{\frac{\frac{1-\cos\theta}{\cancel{2}}}{\frac{1+\cos\theta}{\cancel{2}}}} = \boxed{\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}}$$

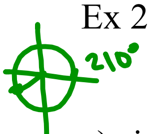
tan$\left(\frac{\theta}{2}\right)$

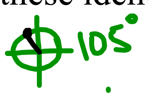
Half Angle Formulas. For all applicable angles θ,

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Ex 2: Use these identities to determine exact values.





a) $\sin(105^\circ) = \sin\left(\frac{210^\circ}{2}\right)$

$$= +\sqrt{\frac{1-\cos(210^\circ)}{2}} = \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}$$

$$= \sqrt{\left(\frac{1+\frac{\sqrt{3}}{2}}{2}\right)\left(\frac{2}{2}\right)} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

b) $\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{\frac{7\pi}{6}}{2}\right)$

$$= \pm\sqrt{\frac{1-\cos\left(\frac{7\pi}{6}\right)}{1+\cos\left(\frac{7\pi}{6}\right)}}$$

$$= -\sqrt{\left(\frac{1-\left(-\sqrt{3}/2\right)}{1+\left(-\sqrt{3}/2\right)}\right)\left(\frac{2}{2}\right)}$$

$$= \boxed{-\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}}$$

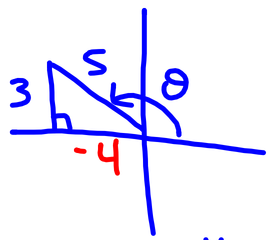
Ex 3: If θ is an obtuse angle and $\sin\theta = \frac{3}{5}$, find the exact value of these using double/half angle identities.

a) $\sin(2\theta)$

$= 2\cos\theta\sin\theta$

$= 2\left(\frac{-4}{5}\right)\left(\frac{3}{5}\right)$

$= \frac{-24}{25}$



$\cos\theta = \frac{-4}{5}$

b) $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

$= \pm\sqrt{\frac{1+\frac{-4}{5}}{2}}$

$= \pm\sqrt{\frac{\frac{1}{5}}{2}} = \pm\sqrt{\frac{1}{10}}$

$\boxed{\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{10}}}$

which is it, + or −?

$0 < \theta < \pi$

$0 < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \frac{\theta}{2}$ in Q1

$\Rightarrow$ choose +

c) $\tan(2\theta)$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\left(\frac{-3}{4}\right)}{1-\left(\frac{-3}{4}\right)^2} = \left(\frac{\frac{-3}{2}}{1-\frac{9}{16}}\right)\left(\frac{16}{16}\right)$

$= \frac{\frac{-3(\cancel{16})8}{\cancel{2}}}{16-9} = \frac{-24}{7}$

Ex 4: Evaluate $\cos\left(\frac{7\pi}{12}\right)$ in two ways, using the half-angle identity and using the difference identity.

① using half-angle id.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\frac{7\pi}{6}}{2}\right)$$

$$= \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}$$

$$= \pm\sqrt{\left(\frac{1+{-\sqrt{3}}/2}{2}\right)\left(\frac{2}{2}\right)}$$

$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\cos\left(\frac{7\pi}{12}\right) = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

which is it, + or - ?

$$\boxed{\cos\left(\frac{7\pi}{12}\right) = \frac{-\sqrt{2-\sqrt{3}}}{2}}$$

② using difference id.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{(9-2)\pi}{12}\right)$$

$$= \cos\left(\frac{9\pi}{12} - \frac{2\pi}{12}\right)$$

$$= \cos\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{-\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{-\sqrt{6}+\sqrt{2}}{4}}$$