3.7 ~ Graphing polar equations

In this lesson you will:

• Graph polar equations by point plotting.

• Use symmetry, zeros and maximum r-values to sketch graphs of polar equations.

• Recognize special polar graphs.
What do these equations represent?

These are all lines!

1. \( \theta = \beta \)  
   (\( \beta \) constant) 
   (radial line through the pole/origin)

2. \( r \cos \theta = a \)  
   (\( a \) constant) 
   Vertical line

3. \( r \sin \theta = b \)  
   (\( b \) constant) 
   Horizontal line
What about these?  

All of these eqns represent circles (that go thru origin):

1. \( r = 2a \cos \theta \)  
   \( r^2 = 2a(r \cos \theta) \iff x^2 + y^2 = 2ax \)

2. \( r = 2b \sin \theta \)  
   Circle centered at \((0, b)\) w/ radius \(b\)

3. \( r = 2a \cos \theta + 2b \sin \theta \)  
   Circle centered at \((a, b)\)  
   w/ radius of \(\sqrt{a^2 + b^2}\)

Example 1:

\[ r = 4 \cos \theta \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \pi/4 )</th>
<th>( \pi/3 )</th>
<th>( 2\pi/3 )</th>
<th>( 3\pi/4 )</th>
<th>( \pi )</th>
<th>( 5\pi/4 )</th>
<th>( 3\pi/2 )</th>
<th>( 7\pi/4 )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>4</td>
<td>2\sqrt{2}</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>-2\sqrt{2}</td>
<td>-2\sqrt{2}</td>
<td>0</td>
<td>2\sqrt{2}</td>
<td>4</td>
</tr>
</tbody>
</table>

\((4, 0) = (-4, \pi)\)

\((2\sqrt{2}, \frac{\pi}{4}) = (-2\sqrt{2}, \frac{5\pi}{4})\)

\((0, \frac{\pi}{2}) = (0, \frac{3\pi}{2})\)

\((2\sqrt{2}, 2\pi) = (\frac{3\pi}{4}, -2\sqrt{2})\)
Symmetry

1. Symmetry with respect to the line $\theta = \pi/2$
   - Replace $(r, \theta)$ with $(r, \pi-\theta)$ or $(-r, \theta)$:
   - If an equivalent equation results, the graph has this type of symmetry.

2. Symmetry with respect to the polar axis ($\theta = 0$):
   - Replace $(r, \theta)$ with $(r, -\theta)$ or $(-r, \pi-\theta)$:
   - If an equivalent equation results, the graph has this type of symmetry.

3. Symmetry with respect to the pole
   - Replace $(r, \theta)$ with $(-r, \theta)$ or $(r, \pi+\theta)$:
   - If an equivalent equation results, the graph has this type of symmetry.

If a polar equation passes a symmetry test, then its graph definitely exhibits that symmetry. However, if a polar equation fails a symmetry test, then its graph may or may not have that kind of symmetry.
Zeros and maximum $r$-values

Other helpful tools in graphing polar equations are knowing the values for $\theta$ for which $|r|$ is maximum and those for which $r = 0$.

Example 2: Graph \( r = \frac{1}{2} + \cos \theta \)

Symmetry:

1. Replace \((r, \theta)\) with \((-r, -\theta)\): 
   \[-r = \frac{1}{2} + \cos(-\theta)\]
   \[-r = \frac{1}{2} + \cos(\theta)\]
   \(\iff \quad r = \frac{1}{2} + \cos \theta\)

2. Replace \((r, \theta)\) with \((r, -\theta)\): 
   \[r = \frac{1}{2} + \cos(\theta)\]
   \(\text{symmetry w.r.t } \theta = 0 \quad (x\text{-axis}) \quad \checkmark \quad r = \frac{1}{2} + \cos \theta\)

3. Replace \((r, \theta)\) with \((-r, \theta)\): 
   \[-r = \frac{1}{2} + \cos \theta\]

\(|r|\) maximum: 
\[r = \frac{1}{2} + \cos \theta \quad \iff \quad r = \frac{1}{2} + \cos \theta \quad \text{max when } \cos \theta = 1 \quad \Rightarrow \quad r \text{ max value } = \frac{3}{2} \quad \text{when } \theta = \pi, 2\pi, \ldots\]

Zero of \(r\):
\[0 = \frac{1}{2} + \cos \theta \quad \iff \quad \cos \theta = \frac{1}{2} \quad \iff \quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}\]

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
<th>$2\pi/3$</th>
<th>$3\pi/4$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1+\sqrt{5}}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1-\sqrt{5}}{2}$</td>
<td>$\frac{-1}{2}$</td>
</tr>
</tbody>
</table>

\(r = \frac{1}{2} + \cos \theta\)

Limaçon
Limaçons

\[ r = a \pm b \cos \theta \]
\[ r = a \pm b \sin \theta \]

\( a, b \) constants
\( a > 0, \ b > 0 \)
(all these picture examples are the cosine varieties)

\[ \frac{a}{b} \geq 2 \]
Convex limaçon

\[ 1 < \frac{a}{b} < 2 \]
Dimpled limaçon

\[ \frac{a}{b} = 1 \]
Cardioid -always passes through pole

\[ \frac{a}{b} < 1 \]
Limaçon with inner loop
Example 3: Graph \( r = 3 \sin 2\theta \)

Symmetry: 
by replacing \((r, \theta) \leftrightarrow (-r, -\theta)\)

\[-r = 3 \sin(-2\theta)\]  
\[-r = -3 \sin(2\theta)\]  
\[r = 3 \sin(2\theta)\]  
Symmetry w.r.t \(\Theta = \frac{\pi}{2}\) (y-axis)

\(|r|\) maximum:

\(r\) is max when \(\sin(2\theta) = \pm 1\) \(\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}\)

\(\Theta = \frac{\pi}{4}, \frac{3\pi}{4}\)

Zero of \(r\):

\(0 = 3 \sin(2\theta)\)

\(\sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi\)

\(\Theta = 0, \frac{\pi}{2}\)

\(r = 3 \sin(2\theta)\)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\theta & 0 & \frac{\pi}{8} & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} \\
\hline
r & 0 & \frac{3\sqrt{2}}{2} & \frac{3\sqrt{3}}{2} & 3 & \frac{3\sqrt{2}}{2} & 0 \\
\hline
\end{array}
\]

\[
\frac{3\sqrt{2}}{2} \approx 2.12 \\
\frac{3\sqrt{3}}{2} \approx 2.6 \\
(-3, \frac{3\pi}{4})
\]
Roses

\[
\begin{align*}
  r &= a \sin(n \theta), \\
  r &= a \cos(n \theta).
\end{align*}
\]

or

If \( n \) is odd, the rose is \( n \)-petalled. If \( n \) is even, the rose is \( 2 \cdot n \)-petalled.

\[\begin{array}{cccc}
  n = 2 & n = 3 & n = 4 & n = 5 \\
  \includegraphics[width=1in]{rose2} & \includegraphics[width=1in]{rose3} & \includegraphics[width=1in]{rose4} & \includegraphics[width=1in]{rose5}
\end{array}\]

No reason to limit ourselves to \( n \) integer:

\[\begin{array}{cccc}
  n = 1/2 & n = 1/3 & n = 2/3 & n = 1/4 \\
  \includegraphics[width=1in]{rose1/2} & \includegraphics[width=1in]{rose1/3} & \includegraphics[width=1in]{rose2/3} & \includegraphics[width=1in]{rose1/4}
\end{array}\]

\[\begin{array}{cccc}
  n = 3/4 & n = 1/5 & n = 2/5 & n = 3/5 \\
  \includegraphics[width=1in]{rose3/4} & \includegraphics[width=1in]{rose1/5} & \includegraphics[width=1in]{rose2/5} & \includegraphics[width=1in]{rose3/5}
\end{array}\]

Or even rational:

\[\begin{array}{ccc}
  n = e & n = \pi & n = \sqrt{2} \\
  \includegraphics[width=1in]{rose_e} & \includegraphics[width=1in]{rose_pi} & \includegraphics[width=1in]{rose_sqrt2}
\end{array}\]