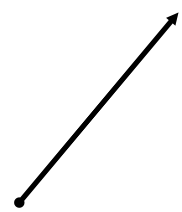
Trig 3.3, 3.4 ~ Vectors

- Represent vectors as directed line segments.
- Perform basic vector operations and represent them graphically.
- Write vectors as a linear combination of unit vectors.
- Find the direction angles of vectors.
- Use vectors to solve real-life problems.
- Find the dot product of two vectors.
- Find the angle between two vectors using the dot product.

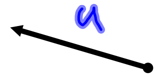
A vector is a directed line segment.

A vector has direction and magnitude independent of the position. location doesn't matter

① ② length

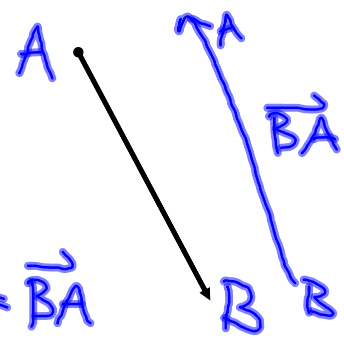


$\vec{u}$



tip (final pt)

tail (initial pt)



$\overrightarrow{AB} = -\overrightarrow{BA}$

The name of a vector:

$\overrightarrow{AB} \neq \overrightarrow{BA}$

$\vec{u}$

Parts of a vector:
- tip
- tail

(note: in book see $\vec{u}$ written as **u**)

Two vectors are the same if they have the same direction and the same magnitude independent of position.

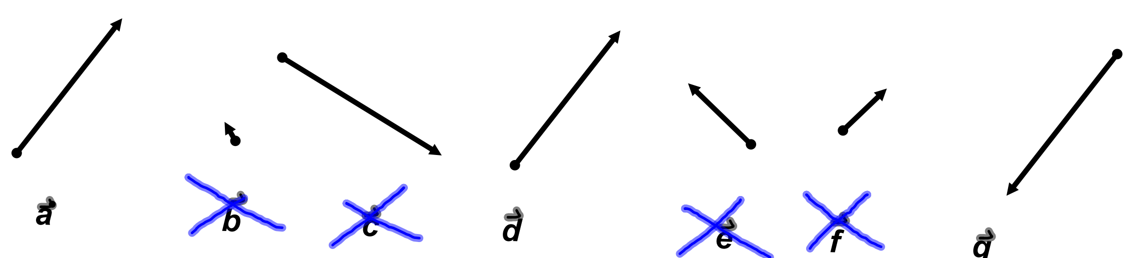
$\vec{a} = \vec{b}$

Opposite vectors have the same magnitude and opposite directions.

$\vec{a}$, $-\vec{a}$

Select the two equivalent vectors:

Select the two opposite vectors:



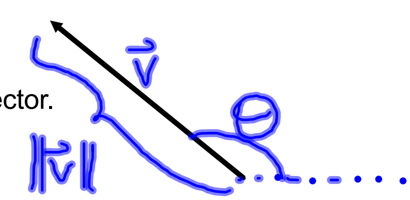
$\vec{a} = \vec{d}$

$\vec{a} = -\vec{g}$ or $\vec{d} = -\vec{g} \Leftrightarrow -\vec{d} = \vec{g}$

For vector ***v***

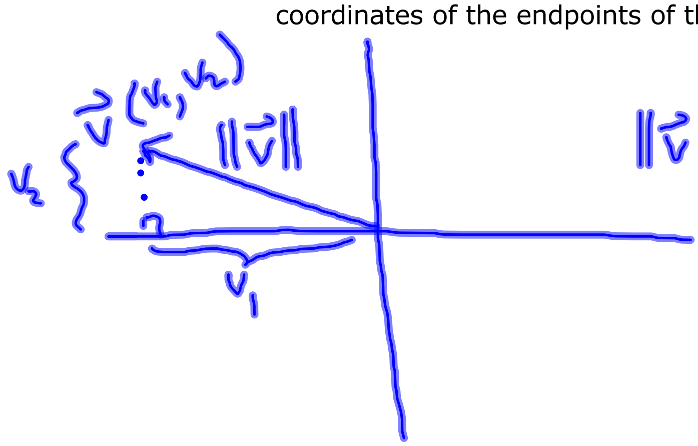
→ ||v|| represents the magnitude or length of the vector.

θ represents the direction angle of the vector.



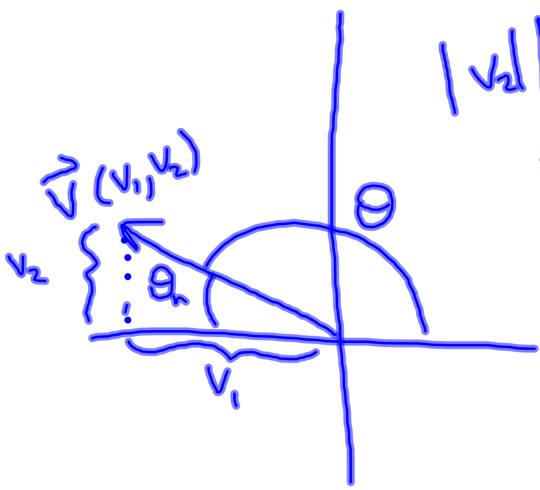
In General:

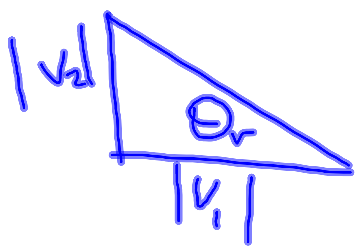
||v|| The magnitude of a vector is found using the Pythagorean Theorem on the coordinates of the endpoints of the vector.



$$\|\vec{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2}}$$

θ The direction angle is found with trigonometry by using arctan to find the <u>reference angle</u>, then placing the angle in the correct quadrant.





$$\tan \theta_r = \frac{|v_2|}{|v_1|}$$

$$\theta_r = \arctan\left|\frac{v_2}{v_1}\right|$$

We will consider vectors in three forms:

1. Described by the coordinates of the tail and the tip.

   v = $\overrightarrow{AB}$ where A has coordinates (-3,-2) and B has (1,5).

2. In standard position, placing the tail on the origin and stating the coordinates of the tip.

   v is $\langle 4, 7 \rangle$ in standard position.

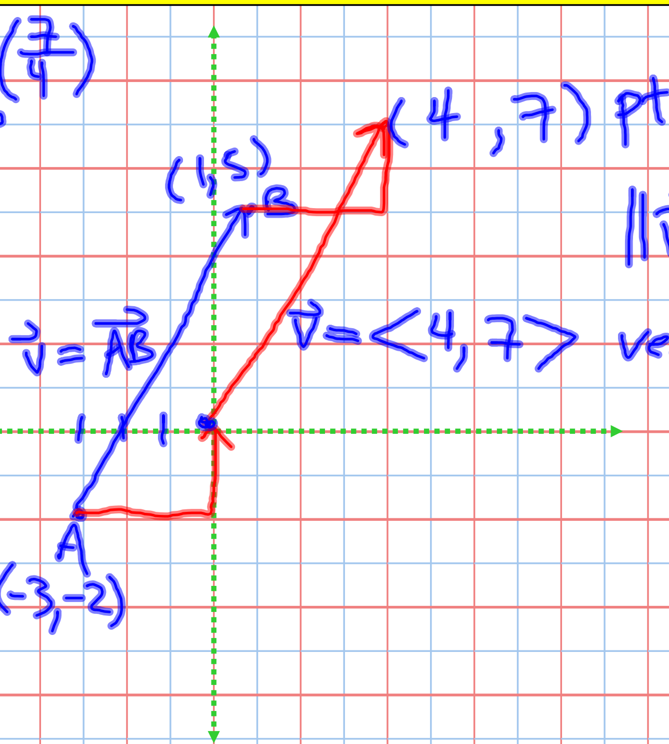
3. Describing the magnitude and direction of the vector.

   v = $\|v\|$ $\langle \cos\theta, \sin\theta \rangle$ where is the angle of the vector in standard position.

   $= \sqrt{65} \langle \cos 60.26°, \sin 60.26° \rangle$

$\theta_r = \theta = \arctan\left(\frac{7}{4}\right)$

$\simeq 60.26°$



$\|\vec{v}\| = \sqrt{4^2+7^2}$

vector $= \sqrt{16+49}$

$= \sqrt{65}$

remember:

(4,7)

$= (r\cos\theta, r\sin\theta)$

$= \|\vec{v}\| (\cos\theta, \sin\theta)$

$\mathbf{u} = \langle 3,2 \rangle$ $\mathbf{v} = \langle -2,1 \rangle$

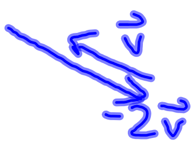
Vector arithmetic:

A vector may be multiplied by a scalar.

$3\mathbf{u} = 3\langle 3,2\rangle = \langle 9,6\rangle$

$-2\mathbf{v} = -2\langle -2,1\rangle$

$= \langle 4,-2\rangle$



note: scalar → constant $\mathbb{R}$ number

Two vectors may be added by adding their components.

$\mathbf{u} + \mathbf{v} = \langle 3,2\rangle + \langle -2,1\rangle = \langle 3+-2, 2+1\rangle$

$= \langle 1,3\rangle$

$\mathbf{u} - 2\mathbf{v} = \langle 3,2\rangle - 2\langle -2,1\rangle$

$= \langle 3,2\rangle + \langle 4,-2\rangle$

$= \langle 3+4, 2+-2\rangle$

$= \langle 7,0\rangle$

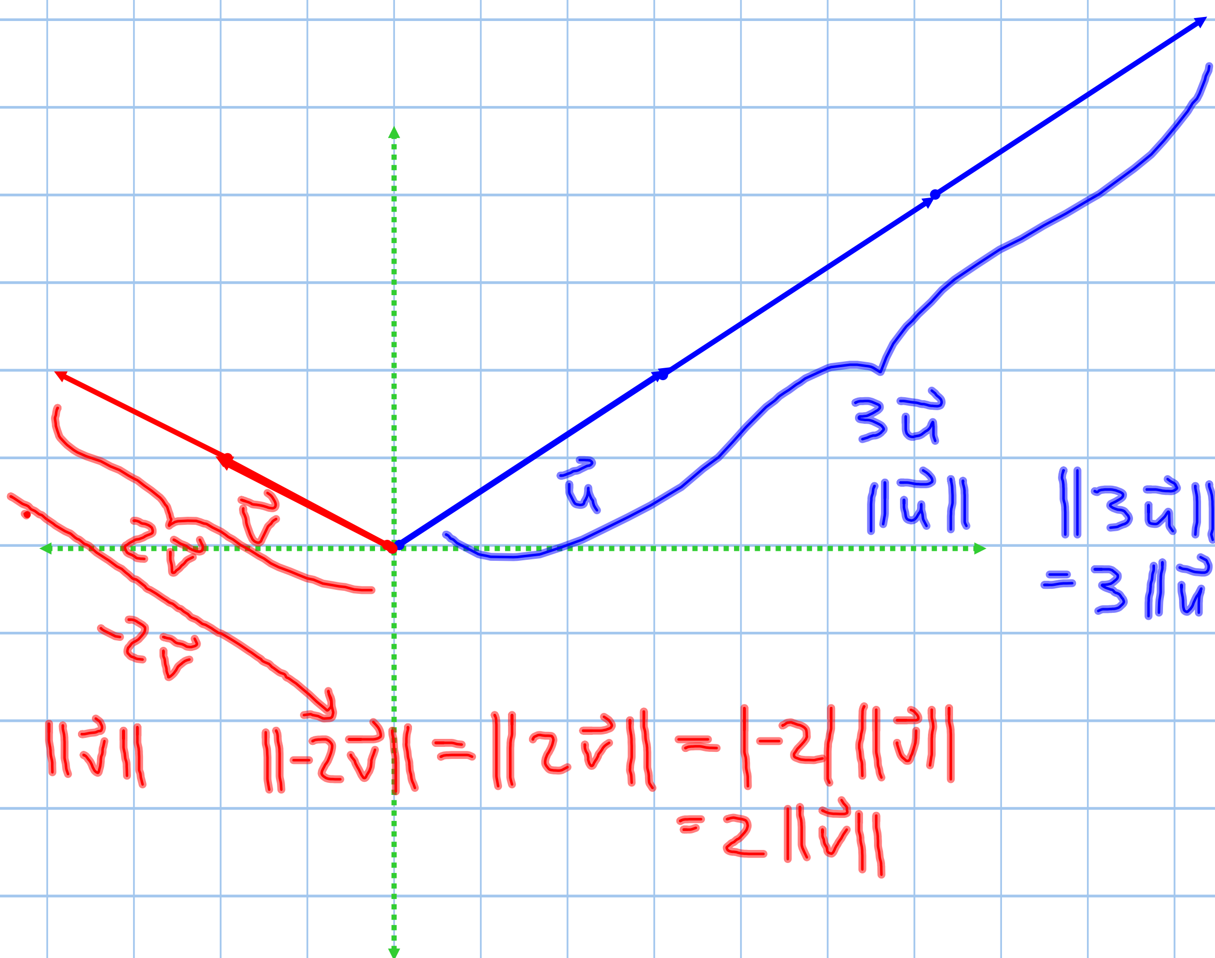
Vector arithmetic is easier to visualize geometrically:

When adding two vectors, place them tail to tip. The resultant is the vector from the tail of the first to the tip of the last.

$\boldsymbol{u} = \langle 3,2 \rangle$ $\boldsymbol{v} = \langle -2,1 \rangle$

$3\,\boldsymbol{u} = 3\langle 3,2\rangle = \langle 9,6\rangle$

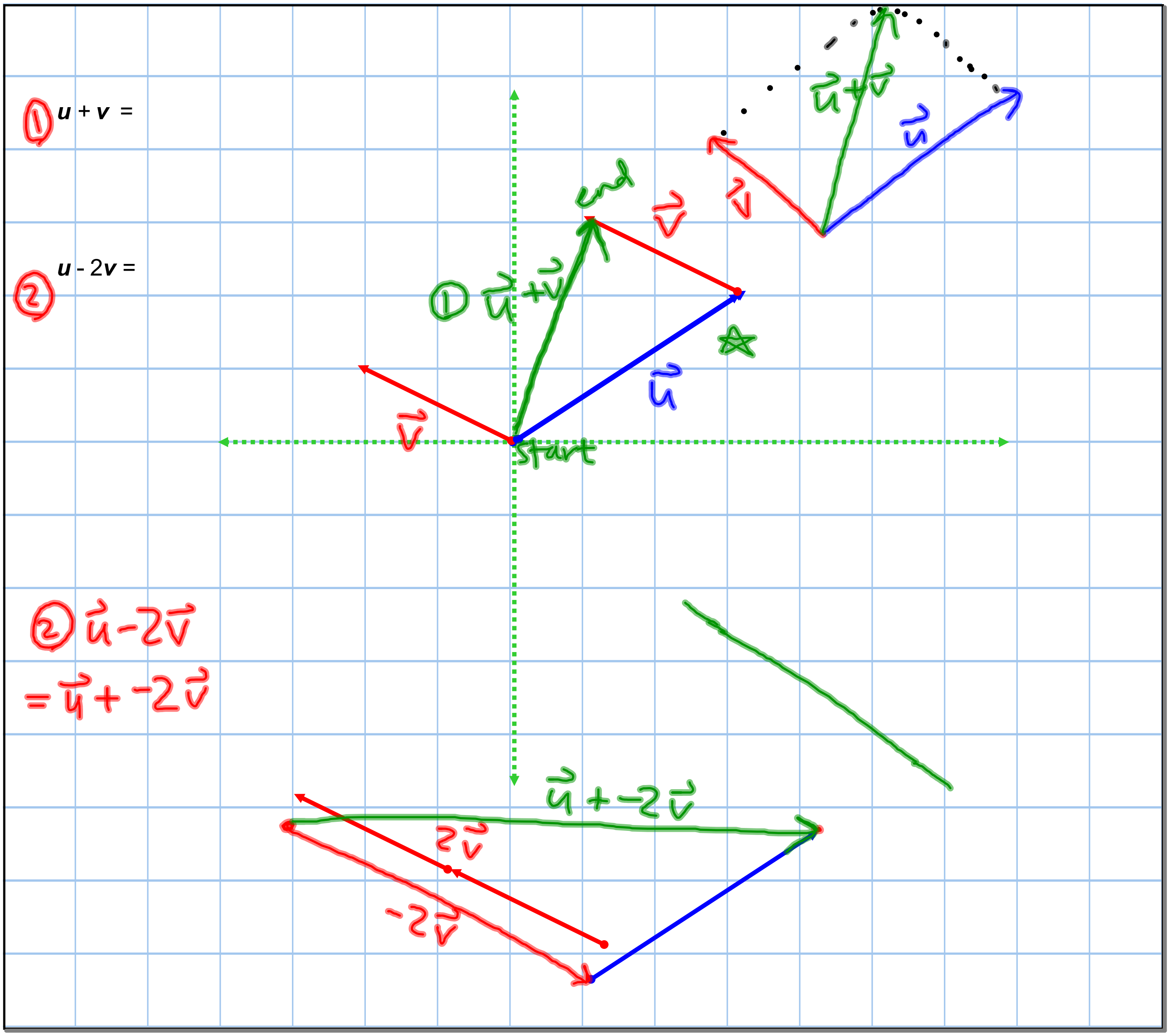
$-2\,\boldsymbol{v} =$



$\|3\vec{u}\| = 3\|\vec{u}\|$

$\|\vec{v}\|$

$\|-2\vec{v}\| = \|2\vec{v}\| = |-2|\,\|\vec{v}\| = 2\|\vec{v}\|$


① u + v =
② u - 2v =
① $\vec{u}+\vec{v}$
end
$\vec{v}$
$\vec{u}$
$\vec{v}$
start
$\vec{u}+\vec{v}$
$\vec{u}$
$\vec{v}$
② $\vec{u}-2\vec{v}$
$=\vec{u}+{}^-2\vec{v}$
$\vec{u}+-2\vec{v}$
$2\vec{v}$
$-2\vec{v}$

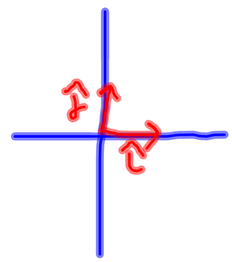
The unit vectors, ***i*** and ***j*** give us one more way to express our vectors.

unit vector $\Rightarrow$ a vector w/ length of 1.

$\|\vec{u}\| = 1$

notation: $\hat{u}$

$\hat{\imath}, \hat{\jmath}$ $\quad \hat{\imath} = \langle 1, 0 \rangle \quad \hat{\jmath} = \langle 0, 1 \rangle$



$\|\hat{\imath}\| = \sqrt{1^2 + 0^2}$
$= 1$

note: $\hat{\imath} = \hat{x}$
$\hat{\jmath} = \hat{y}$

$\vec{u} = \langle 5, 4 \rangle = 5\langle 1, 0 \rangle + 4\langle 0, 1 \rangle$
$= 5\hat{\imath} + 4\hat{\jmath}$

5 in x dir

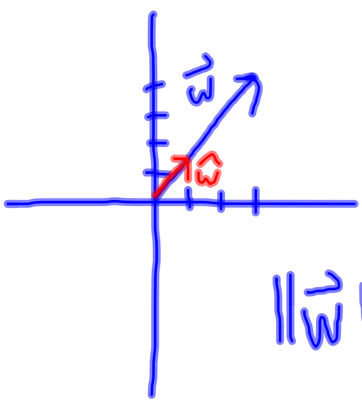
4 in y dir

$\vec{v} = \langle -3, 0 \rangle$
$= -3\hat{\imath}$

$\vec{w} = \langle -2, 10 \rangle = -2\hat{\imath} + 10\hat{\jmath}$

Unit Vectors

Ex $\vec{w} = \langle 3, 4 \rangle$ find $\hat{w}$



$$\boxed{\hat{w} = \frac{\vec{w}}{\|\vec{w}\|}}$$

Pf $\|\hat{w}\| = \left\| \frac{\vec{w}}{\|\vec{w}\|} \right\|$

Let $c = \|\vec{w}\|$. $c \in \mathbb{R}$.

$\|\hat{w}\| = \frac{\|\vec{w}\|}{c} = \frac{\|\vec{w}\|}{\|\vec{w}\|}$

$= 1$ //

$\|\vec{w}\| = \sqrt{3^2 + 4^2}$

$= \sqrt{25} = 5$

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{w}}{5} = \frac{\langle 3, 4 \rangle}{5} = \boxed{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$$

Vectors: The dot product. (dot product of 2 vecs = scalar)

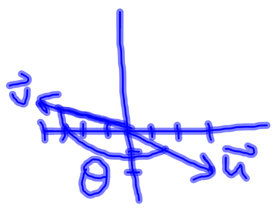
The dot product of two vectors provides a formula which will help find the angle between two vectors.

Defn ① algebraic $\vec{u}\cdot\vec{v} = \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle$
$= u_1v_1 + u_2v_2$

② geometric $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$
(θ angle between $\vec{u}$ & $\vec{v}$)

So given vectors $\boldsymbol{u} = \langle 3,-2\rangle$ and $\boldsymbol{v} = \langle -4,1\rangle$

Find the dot product: $\vec{u}\cdot\vec{v} = \langle 3,-2\rangle \cdot \langle -4,1\rangle$
$= 3(-4) + -2(1)$
$= \underline{-14} \Rightarrow \theta$ in Q2



The cosine of the angle between the two vectors is the dot product divided by the product of the magnitudes of the two vectors.

$$\cos\phi = \frac{\boldsymbol{u}\circ\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} = \left(\frac{\vec{u}}{\|\vec{u}\|}\right)\cdot\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \hat{u}\cdot\hat{v}$$

Find the angle between the two vectors above: $\vec{u} = \langle 3,-2\rangle$, $\vec{v} = \langle -4,1\rangle$

$$\cos\theta = \frac{-14}{\sqrt{13}\sqrt{17}}$$

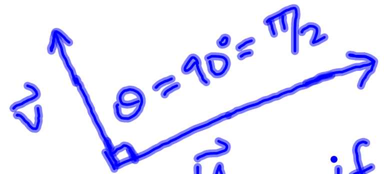
$$\|\vec{u}\| = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$$

$$\|\vec{v}\| = \sqrt{(-4)^2+1^2} = \sqrt{17}$$

$$\boxed{\theta = \arccos\left(\frac{-14}{\sqrt{13}\sqrt{17}}\right)}$$

$\theta = 160.3^\circ$

Orthogonal vectors: If two vectors are perpendicular to each other they are said to be orthogonal. What would the cosine of two orthogonal vectors be?



$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\ \|\vec{v}\| \cos\theta$$

if $\cos\theta = 0$

$$\Rightarrow \vec{u}\cdot\vec{v} = 0$$

$$\boxed{\vec{u}\cdot\vec{v} = 0 \iff \vec{u} + \vec{v} \text{ are orthogonal}}$$

Orthogonal or not?

1. $\vec{u} = \langle 3,-2 \rangle$ and $\vec{v} = \langle 1,4 \rangle$
2. $\vec{u} = \langle 4,-6 \rangle$ and $\vec{v} = \langle -3,-2 \rangle$

(1) $\vec{u}\cdot\vec{v} = 3(1) + -2(4)$
$= 3 - 8 = -5 \neq 0$
$\Rightarrow \vec{u}$ not orthog. to $\vec{v}$

$\vec{u}\cdot\vec{v} = 4(-3) + -6(-2)$
$= -12 + 12 = 0 \Rightarrow \vec{u}$ & $\vec{v}$ $\perp$

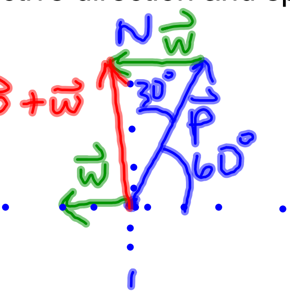
Application problem 1: - flying an airplane.

A plane is flying N 30º E at 400 mph and the wind is blowing west at 40 mph.
What is the effective direction and speed of the plane?

$\theta = 60^\circ$ $\quad \|\vec{p}\| = 400$

$\theta = 180^\circ$ $\quad \|\vec{w}\| = 40$

Draw a picture.



$$\vec{p} = \|\vec{p}\| \langle \cos\theta, \sin\theta \rangle$$
$$= 400 \langle \cos 60^\circ, \sin 60^\circ \rangle$$
$$= 400 \langle 1/2, \sqrt{3}/2 \rangle$$

Place your vectors for proper addition.

Remember the resultant is from the tail of the first to the tip of the second.

$$= \langle 200, 200\sqrt{3} \rangle$$

$$\vec{p} = 200\hat{i} + 200\sqrt{3}\hat{j}$$
$$\vec{w} = -40\hat{i}$$
$$\vec{p}+\vec{w} = (200 + -40)\hat{i} + 200\sqrt{3}\hat{j}$$
$$= 160\hat{i} + 200\sqrt{3}\hat{j}$$
$$= \boxed{\langle 160, 200\sqrt{3} \rangle} \text{ mph}$$

$$\|\vec{p}+\vec{w}\|$$

$$\vec{w} = \|\vec{w}\| \langle \cos\theta, \sin\theta \rangle$$
$$= 40 \langle \cos 180^\circ, \sin 180^\circ \rangle$$
$$= 40 \langle -1, 0 \rangle$$
$$= \langle -40, 0 \rangle = -40\hat{i}$$

When computed on a calculator, the magnitude of the velocity of the plane is 381.58 mph.
The direction is N 25º E.

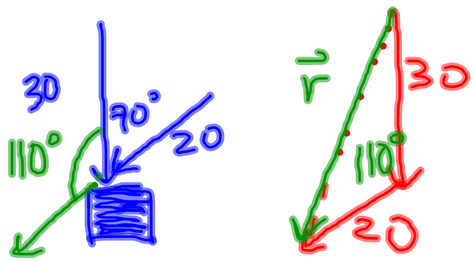
Application problem 2 forces acting on an object:

Two forces are pushing on an object, one exerts 30 lbs of pressure and a second exerts 20 lbs of pressure. The angle between the two forces is 70º. What is the resultant force on the object?

note:
30 lbs = magnitude of that force vector

$\vec{r}$ = resulting force vector

want $\|\vec{r}\| = ?$



$$\|\vec{r}\|^2 = 30^2 + 20^2 - 2(30)(20)\cos 110°$$

$$\|\vec{r}\| = \sqrt{30^2 + 20^2 - 2(30)(20)\cos 110°}$$

When computed on a calculator he resultant is 41.36 lb.

Application 3: Using a ramp to lift heavy objects.

A 500-lb rock is being wheeled up a 30 degree ramp. What force is necessary to keep it from rolling back down the ramp? What is the weight the ramp is actually supporting?

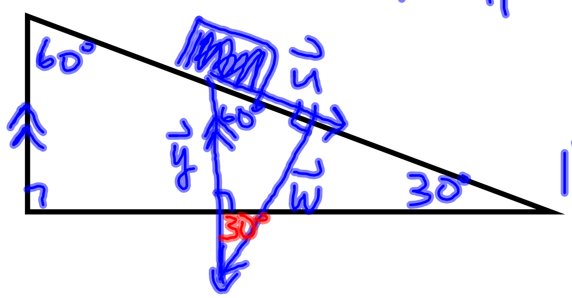
note: 500 lb (already includes gravity)

$\|\vec{y}\| = 500$

$\vec{w} \perp$ to ramp

$\|\vec{w}\| =$ weight of object that ramp is supporting

$\|\vec{u}\| =$ magnitude of force required to keep weight in same place



$\cos 30° = \frac{\|\vec{w}\|}{500}$

$\Leftrightarrow \|\vec{w}\| = 500\cos 30°$ (lbs)

$\|\vec{u}\| = 500\sin 30°$ (lbs)

$\|\vec{w}\| = 250\sqrt{3}$ lbs ≈ 433 lbs

$\|\vec{u}\| = 250$ lbs