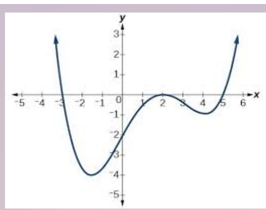
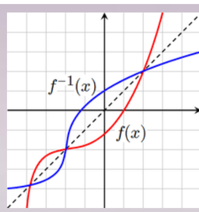

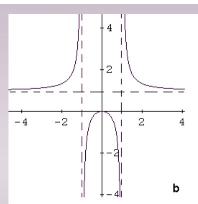

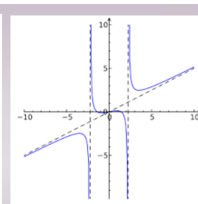
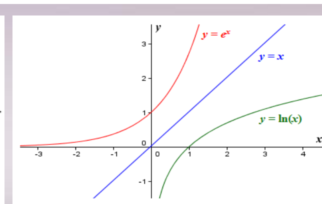

Math 1050 ~ College Algebra

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

9 Real Zeros of Polynomials

Learning Objectives

- Find possible (potential) rational zeros using the Rational Zeros Theorem.
- Find real zeros of a polynomial and their multiplicities.

We are now ready to determine the rational roots of a polynomial.

Rational Zeros Theorem

If $f(x)$ is a polynomial that has integer coefficients, every rational zero of $f(x)$ has the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.

Ex 1: Use the Rational Zeros Theorem to determine the possible roots of these functions.

a) $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

constant: 3

leading coefficient: 2

p factors of 3 are $\pm 1, \pm 3$

q factors of 2 are $\pm 1, \pm 2$

possible rational zeros of $f(x)$ are $\frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 2}, \frac{\pm 3}{\pm 1}, \frac{\pm 3}{\pm 2}$

simplifies to possible factors $\boxed{\pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}}$

b) $g(x) = 3x^3 + 3x^2 - 11x - 10$

const: -10

l.c.: 3

p $\pm 1, \pm 2, \pm 5, \pm 10$

q $\pm 1, \pm 3$

possible rational zeros of $g(x)$

$\boxed{\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{5}{3}, \pm \frac{10}{3}}$

This rule may further help you in eliminating some of the options when determining the roots of a polynomial.

Descartes Rule of Signs

Given a polynomial function with real coefficients and a constant term not zero:

- The number of positive real roots is equal to the number of variations in signs of *f(x)* or less than that by an even number.
- The number of negative real roots is equal to the number of variations in signs of *f(-x)* or less than that by an even number.

Ex 2: Determine how many positive and negative roots these functions are likely to have.

a) $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

2 variations of sign

$\Rightarrow$ 2 or 0 pos. roots

$f(-x) = 2x^4 - x^3 - 7x^2 + 3x + 3$

2 variations of sign

$\Rightarrow$ 2 or 0 neg. roots

b) $g(x) = 3x^3 + 3x^2 - 11x - 10$

1 change/variation in sign

$\Rightarrow$ exactly 1 pos. root

$g(-x) = -3x^3 + 3x^2 + 11x - 10$

2 changes in sign

$\Rightarrow$ 2 or 0 neg. roots

Ex 3: Find all zeros for these functions.

a) $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

possible roots: $\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}$

2 or 0 pos roots

2 or 0 neg roots

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -7 & -3 & 3 \\ & & -2 & 1 & 6 & -3 \\ \hline & 2 & -1 & -6 & 3 & 0 \end{array}$$

remainder

$\Rightarrow -1$ is root/zero of $f(x)$

$\Leftrightarrow (x+1)$ is a factor of $f(x)$

$f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

$f(x) = (x+1)(2x^3 - x^2 - 6x + 3)$

$$\begin{array}{r|rrrr} -3 & 2 & -1 & -6 & 3 \\ & & -6 & 21 & -45 \\ \hline & 2 & -7 & 15 & -42 \end{array}$$

$\Rightarrow -3$ is NOT a root of $f(x)$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -1 & -6 & 3 \\ & & -1 & 1 & \frac{5}{2} \\ \hline & 2 & -2 & -5 & (\neq 0) \end{array}$$

$\Rightarrow -\frac{1}{2}$ is NOT a zero of $f(x)$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & -1 & -6 & 3 \\ & & -3 & 6 & 0 \\ \hline & 2 & -4 & 0 & 3 \end{array}$$

$\Rightarrow -\frac{3}{2}$ is NOT a zero of $f(x)$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & -6 & 3 \\ & & 1 & 0 & -3 \\ \hline & 2 & 0 & -6 & 0 \end{array}$$

$\Rightarrow \frac{1}{2}$ is a root of $f(x)$

$\Leftrightarrow (x - \frac{1}{2})$ is factor

$f(x) = (x+1)(x-\frac{1}{2})(2x^2 - 6)$

$f(x) = (x+1)(x-\frac{1}{2})(2)(x^2 - 3)$

$f(x) = 2(x+1)(x-\frac{1}{2})(x-\sqrt{3})(x+\sqrt{3})$

zeros of $f(x)$:

$x = -1, \frac{1}{2}, \sqrt{3}, -\sqrt{3}$

b) $g(x) = 3x^3 + 3x^2 - 11x - 10$

possible roots: $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 5, \pm\frac{5}{3}, \pm 10, \pm\frac{10}{3}$

1 pos root

2 or 0 neg roots

$$\begin{array}{r|rrrr} 1 & 3 & 3 & -11 & -10 \\ & & 3 & 6 & -5 \\ \hline & 3 & 6 & -5 & -15 \end{array}$$

$\Rightarrow 1$ is NOT a zero of $g(x)$

$$\begin{array}{r|rrrr} -2 & 3 & 3 & -11 & -10 \\ & & -6 & 6 & 10 \\ \hline & 3 & -3 & -5 & 0 \end{array}$$

$\Rightarrow -2$ is zero of $g(x)$

$\Leftrightarrow (x+2)$ is factor of $g(x)$

$g(x) = (x+2)(3x^2 - 3x - 5)$

Find zeros of $g(x)$:

$(x+2)(3x^2 - 3x - 5) = 0$

$x + 2 = 0$ or $3x^2 - 3x - 5 = 0$

$\boxed{x = -2}$

$x = \frac{3 \pm \sqrt{9 - 4(3)(-5)}}{2(3)}$

$\boxed{x = \frac{3 \pm \sqrt{69}}{6}}$

Multiplicity of Roots

A factor $(x-a)^k$, $k>1$, yields <u>repeated zero</u> $x = a$ of <u>multiplicity</u> k.

Ex 4: Determine the roots and state the multiplicity of each. Write in factored form. $f(x) = x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8$

Descartes Rule of Signs: ① $\cancel{||||}$ 5 or 3 or 1 pos. roots

② $f(-x) = -x^5 - 8x^4 - 25x^3 - 38x^2 - 28x - 8$

$\Rightarrow$ 0 neg. roots

Possible Rational Roots/Zeros: $\dfrac{\text{factors of } 8}{\text{factors of } 1}$

1, 2, 4, 8

$$\begin{array}{r|rrrrrr} 1 & 1 & -8 & 25 & -38 & 28 & -8 \\ & & 1 & -7 & 18 & -20 & 8 \\ \hline & 1 & -7 & 18 & -20 & 8 & \boxed{0} \end{array}$$

$\Rightarrow f(x) = (x-1)(x^4 - 7x^3 + 18x^2 - 20x + 8)$

$$\begin{array}{r|rrrrr} 1 & 1 & -7 & 18 & -20 & 8 \\ & & 1 & -6 & 12 & -8 \\ \hline & 1 & -6 & 12 & -8 & \boxed{0} \end{array}$$

$\Rightarrow f(x) = (x-1)(x-1)(x^3 - 6x^2 + 12x - 8)$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 12 & -8 \\ & & 1 & -5 & 7 \\ \hline & 1 & -5 & 7 & -1 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\Rightarrow f(x) = (x-1)^2(x-2)(x^2 - 4x + 4)$

$$\begin{array}{r|rrr} 2 & 1 & -4 & 4 \\ & & 2 & -4 \\ \hline & 1 & -2 & 0 \end{array}$$

$\Rightarrow f(x) = (x-1)^2(x-2)(x-2)(x-2)$

$\Leftrightarrow f(x) = (x-1)^2(x-2)^3$

root	multiplicity
1	2
2	3