Math 1050 ~ College Algebra

9 Real Zeros of Polynomials

Learning Objectives

- Find possible (potential) rational zeros using the Rational Zeros Theorem.
- Find real zeros of a polynomial and their multiplicities.

\[
\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}
\]

\[
\sum_{k=1}^{m} k = \frac{m(m + 1)}{2}
\]

\[
\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}
\]
We are now ready to determine the rational roots of a polynomial.

**Rational Zeros Theorem**

If \( f(x) \) is a polynomial that has integer coefficients, every rational zero of \( f(x) \) has the form \( \frac{p}{q} \), where \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient.

Ex 1: Use the Rational Zeros Theorem to determine the possible roots of these functions.

a) \( f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3 \)  
   \[ \text{constant: 3} \]  
   \[ \text{leading coefficient: 2} \]  
   \( \text{factors of 3 are } \pm 1, \pm 3 \)  
   \( \text{factors of 2 are } \pm 1, \pm 2 \)  
   \( \frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 2}, \frac{\pm 3}{\pm 1}, \frac{\pm 3}{\pm 2} \)  
   possible rational zeros of \( f(x) \) are \( \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2} \)  
   simplifies to possible factors \( \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2} \)

b) \( g(x) = 3x^3 + 3x^2 - 11x - 10 \)  
   \[ \text{constant: -10} \]  
   \[ \text{l.c.: 3} \]  
   \( \text{factors of 10 are } \pm 1, \pm 2, \pm 5, \pm 10 \)  
   \( \pm 1, \pm 2, \pm 5, \pm 10 \)  
   possible rational zeros of \( g(x) \) are \( \pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm 5, \pm \frac{10}{3} \)  
   \( \pm \frac{1}{3}, \pm \frac{2}{3}, \pm 5, \pm \frac{10}{3} \)
This rule may further help you in eliminating some of the options when determining the roots of a polynomial.

**Descartes Rule of Signs**

Given a polynomial function with real coefficients and a constant term not zero:
- The number of positive real roots is equal to the number of variations in signs of \( f(x) \) or less than that by an even number.
- The number of negative real roots is equal to the number of variations in signs of \( f(-x) \) or less than that by an even number.

Ex 2: Determine how many positive and negative roots these functions are likely to have.

- a) \( f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3 \)  
  \[\text{2 variations of sign} \Rightarrow 2 \text{ or 0 pos. roots} \]  
  \( f(-x) = 2x^4 - x^3 - 7x^2 + 3x + 3 \)  
  \[\text{2 variations of sign} \Rightarrow 2 \text{ or 0 neg. roots} \]

- b) \( g(x) = 3x^3 + 3x^2 - 11x - 10 \)  
  \[\text{1 change/variation in sign} \Rightarrow \text{exactly 1 pos. root} \]  
  \( g(-x) = -3x^3 + 3x^2 + 11x - 10 \)  
  \[\text{2 changes in sign} \Rightarrow 2 \text{ or 0 neg. roots} \]
Ex 3: Find all zeros for these functions.

a) \( f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3 \)

Possible roots:
- \( \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2} \)
- \( \pm \frac{3}{2} \) (since 3 is a factor of the constant term)
- \( 2 \) or 0 pos roots
- 2 or 0 neg roots

\[
\begin{array}{c|cccc}
-1 & 2 & 1 & -7 & -3 \\
 & 2 & 1 & 6 & -3 \\
\hline
 & 2 & -1 & -6 & 3 \\
\end{array}
\]

\( -1 \) is root/zero of \( f(x) \)

\( (x+1) \) is a factor of \( f(x) \)

\( f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3 \)

\( f(x) = (x+1)(2x^3 - 6x + 3) \)

\[ \begin{array}{c|cccc}
\frac{1}{2} & 2 & -1 & -6 & 3 \\
 & 2 & -1 & 3 & 0 \\
\hline
 & 2 & -2 & -5 & (\neq 0) \\
\end{array} \]

\( \frac{1}{2} \) is NOT a root of \( f(x) \)

\( -\frac{3}{2} \)

\[ \begin{array}{c|cccc}
\frac{1}{2} & 2 & -1 & -6 & 3 \\
 & 2 & -1 & 6 & 0 \\
\hline
 & 2 & -9 & 0 & 3 \\
\end{array} \]

\( -\frac{3}{2} \) is NOT a root of \( f(x) \)

\[
\begin{array}{c|cccc}
\frac{1}{2} & 2 & -1 & -6 & 3 \\
 & 2 & -1 & -6 & 3 \\
\hline
 & 2 & 0 & -6 & 0 \\
\end{array}
\]

\( \frac{1}{2} \) is a root of \( f(x) \)

\( x - \frac{1}{2} \) is a factor

\[
\begin{align*}
\text{zeros of } f(x): \\
x = -1, \frac{1}{2}, \sqrt{3}, -\sqrt{3}
\end{align*}
\]

b) \( g(x) = 3x^3 + 3x^2 - 11x - 10 \)

Possible roots:
- \( \pm 1, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{5}{3}, \pm 10 \)
- \( \frac{2}{3} \) (since 3 is a factor of the constant term)
- 1 pos root
- 2 or 0 neg roots

\[
\begin{array}{c|cccc}
1 & 3 & 3 & -11 & -10 \\
 & 3 & 6 & -5 & -15 \\
\hline
 & 3 & 6 & -5 & 0 \\
\end{array}
\]

\( 1 \) is NOT a root of \( g(x) \)

\( -2 \)

\[ \begin{array}{c|cccc}
3 & 3 & -11 & -10 \\
 & -3 & 6 & 0 \\
\hline
 & 3 & -3 & -5 & 0 \\
\end{array} \]

\( -2 \) is NOT a root of \( g(x) \)

\( (x+2) \) is a factor of \( g(x) \)

\( g(x) = (x+2)(3x^2 - 3x - 5) \)

Find zeros of \( g(x) \):

\( (x+2)(3x^2 - 3x - 5) = 0 \)

\( x+2 = 0 \) or \( 3x^2 - 3x - 5 = 0 \)

\( x = 3 \pm \sqrt{9 - 4(3)(-5)} \)

\( x = 3 \pm \sqrt{69} \)

\( x = \frac{3 \pm \sqrt{69}}{6} \)
Multiplicity of Roots

A factor \((x-a)^k\), \(k>1\), yields repeated zero \(x = a\) of multiplicity \(k\).

Ex 4: Determine the roots and state the multiplicity of each. Write in factored form. \(f(x) = x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8\)

Descartes Rule of Signs: ① No. of pos. roots ≤ 5 or 3 or 1

② \(f(-x) = -x^5 - 8x^4 - 25x^3 - 38x^2 - 28x - 8\) → 0 neg. roots

Possible Rational Roots/Roots: factors of 8 / factors of 1

\(1, \frac{1}{2}, 4, 8\)

\[\begin{array}{c|cccc}
1 & -8 & 25 & -38 & 28 - 8 \\
1 & -7 & 18 & -20 & 8 \\
\hline
1 & -7 & 18 & -20 & 8 \quad \text{(0)}
\end{array}\]

\(\Rightarrow f(x) = (x+1)(x-1)(x^3 - 7x^2 + 18x - 8)\)

\[\begin{array}{c|cccc}
1 & -6 & 12 & -8 \\
1 & -5 & 7 & 1 \\
\hline
1 & -5 & 7 & -1
\end{array}\]

\(\Rightarrow f(x) = (x+1)^2(x-2)(x^2 - 4x + 4)\)

\[\begin{array}{c|cccc}
2 & -4 & 4 \\
2 & -2 & 0 \\
\hline
1 & -4 & 4 & 0
\end{array}\]

\(\Rightarrow f(x) = (x-1)^2(x-2)^2(x-2)\)

\(\Rightarrow f(x) = (x-1)^2(x-2)^3\)

\(\begin{array}{c|c}
\text{root} & \text{multiplicity} \\
1 & 2 \\
2 & 3
\end{array}\)