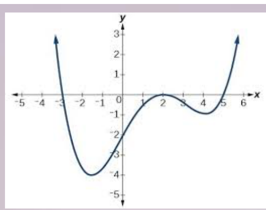

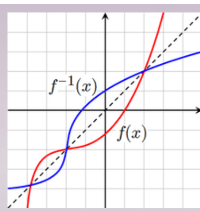

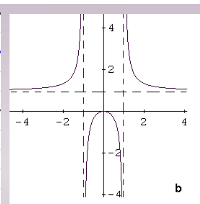

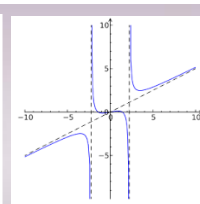

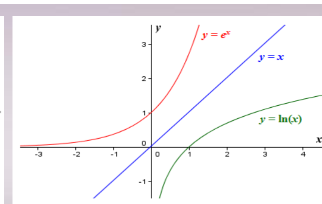

Math 1050 ~ College Algebra

24 Matrix Arithmetic

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Find the sum and difference of two matrices.
- Find the scalar multiple of a matrix.
- Find the product of two matrices.

Definition of a Matrix

A matrix is an array of numbers with m rows and n columns. The size of a matrix is described as $\boxed{m \times n}$. This is called the dimension of a matrix.

Ex 1: Determine the dimension of each matrix.

$$A = \begin{bmatrix} 1 & 0 & -5 \\ -2 & 5 & 0 \\ 1 & 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 7 & 3 & 6 \\ 0 & 2 & -1 & 4 \\ 3 & 5 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -3 & 5 & 1 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

3x3 matrix (square matrix) m=n 3x4 1x4 3x1

$$E = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 2 & 5 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix} \qquad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ identity matrix

3x3 3x2 3x3

The elements of a matrix are called entries. An $m \times n$ matrix has $m \cdot n$ entries. Two matrices are equal only if they are the same dimension and each corresponding element is equal.

Ex 2: If $\begin{bmatrix} x & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 4 & y \end{bmatrix}$, solve for x and y.

2x2 2x2

x=-2 and y=3

Operations with Matrices

Matrix Addition/Subtraction $\Longrightarrow A \pm B$

If two matrices are the same dimension, then addition or subtraction may be accomplished by adding/subtracting the corresponding entries of the two matrices.

$$A = \begin{bmatrix} 1 & 0 & -5 \\ -2 & 5 & 0 \\ 1 & 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 & 3 & 6 \\ 0 & 2 & -1 & 4 \\ 3 & 5 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 5 & 1 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

3x3 3x4 1x4 3x1

$$F = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 2 & 5 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3x3 3x2 3x3

identity matrix

zero matrix 2x3

Ex 2: Determine which matrices from example 1 can be added or subtracted, then perform each operation on them (Exclude the Identity matrix.)

A and F

① $A+F = \begin{bmatrix} 1+1 & 0+3 & -5+2 \\ -2+0 & 5+-1 & 0+0 \\ 1+2 & 4+5 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -3 \\ -2 & 4 & 0 \\ 3 & 9 & 4 \end{bmatrix}$

② $A-F = \begin{bmatrix} 1-1 & 0-3 & -5-2 \\ -2-0 & 5-(-1) & 0-0 \\ 1-2 & 4-5 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -7 \\ -2 & 6 & 0 \\ -1 & -1 & 0 \end{bmatrix}$

Scalar Multiplication $\Rightarrow \boxed{kA}$ k= constant

A scalar is a real number. Multiplication of a matrix by a scalar, k is accomplished by multiplying each entry by that scalar.

Ex 3: Perform these operations.

a) $5C = 5\begin{bmatrix} -3 & 5 & 1 & 8 \end{bmatrix}$

$= \begin{bmatrix} -15 & 25 & 5 & 40 \end{bmatrix}$

b) $-3G = -3\begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -3 & 0 \\ 6 & -12 \\ -9 & 3 \end{bmatrix}$

c) $2\boldsymbol{I} - 3A$

$= 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -5 \\ -2 & 5 & 0 \\ 1 & 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -15 \\ -6 & 15 & 0 \\ 3 & 12 & 6 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 15 \\ 6 & -13 & 0 \\ -3 & -12 & -4 \end{bmatrix}$

Properties of Matrix Addition

- **Commutative Property:** For all $m \times n$ matrices, $A + B = B + A$ (we can re-order)
- **Associative Property:** For all $m \times n$ matrices, $(A+B)+C = A+(B+C)$ (re-group)
- **Identity Property:** For all $m \times n$ matrices, $A + \mathbf{0} = \mathbf{0} + A = A$ 0 = a zero matrix
- **Inverse Property:** Every $m \times n$ matrix A has a unique **additive inverse**, denoted $-A$, such that $A + (-A) = (-A) + A = \mathbf{0}$

Ex 4: Write an additive identity matrix and an additive inverse matrix for each of these.

a) $G = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ additive identity matrix

$-G = \begin{bmatrix} -1 & 0 \\ 2 & -4 \\ -3 & 1 \end{bmatrix}$ additive inverse of G

b) $C = \begin{bmatrix} -3 & 5 & 1 & 8 \end{bmatrix}$

$0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$-C = \begin{bmatrix} 3 & -5 & -1 & -8 \end{bmatrix}$ additive inverse for C

Multiplying Matrices

To multiply an $m \times p$ and $p \times n$ matrix, note that the number of columns in the first matrix must be equal to the number of rows in the second matrix. The result will be an $m \times n$ matrix.

Let's begin by demonstrating what it means to multiply a row of one matrix by a column of another.

For the matrices $A = \begin{array}{c} R_1 \\ \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \end{array}$ $\quad B = \begin{array}{cc} C_1 & \\ \multicolumn{2}{c}{\begin{bmatrix} 5 & -6 \\ 7 & 0 \end{bmatrix}} \end{array}$ $\quad R_1$ will denote row 1 of A

and C_1 will denote column 1 of B.

To multiply R_1 by C_1 means this: $R_1C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 1(5) + 2(7) = 19$

Ex 5: Determine each of these for A and B above.

a) $R_1C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \end{bmatrix}$
$= 1(-6) + 2(0)$
$= -6 + 0 = -6$

b) $R_2C_1 = \begin{bmatrix} 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$
$= 4(5) + 3(7)$
$= 20 + 21 = 41$

c) $R_2C_2 = \begin{bmatrix} 4 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \end{bmatrix}$
$= 4(-6) + 3(0)$
$= -24 + 0 = -24$

The result is $AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$

$(2\times 2)(2\times 2)$

Ex 6: a) Write the matrix AB for Ex 5. b) Compute BA for the same matrices.

$AB = \begin{bmatrix} 19 & -6 \\ 41 & -24 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -6 \\ 7 & 0 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 5 & -6 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 5(1)+-6(4) & 5(2)+-6(3) \\ 7(1)+0(4) & 7(2)+0(3) \end{bmatrix}$

$2\times 2 \quad 2\times 2$

$= \begin{bmatrix} -19 & -8 \\ 7 & 14 \end{bmatrix}$

$\Rightarrow AB \neq BA$

multiplication of matrices is NOT commutative, i.e. order of multiplication matters.

Matrix Multiplication: Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Let R_i denote the i^{th} row of A and C_j denote the j^{th} column of B. The **product of A and B**, denoted AB, is the matrix defined by

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_n \\ R_2C_1 & R_2C_2 & \cdots & R_2C_n \\ \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_n \end{bmatrix}$$

Ex 7: Perform multiplication on these.

a) $\begin{bmatrix} -3 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

1×3 $\quad 3\times1$

(get back 1×1)

$= \begin{bmatrix} -3(2)+4(5)+0(1) \end{bmatrix}$

$= \begin{bmatrix} 14 \end{bmatrix}$

b) $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}^2$

$(2\times2)(2\times2)$

$= \begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} 5(5)+2(3) & 5(2)+2(-1) \\ 3(5)+-1(3) & 3(2)+-1(-1) \end{bmatrix}$

$= \begin{bmatrix} 31 & 8 \\ 12 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -1 & 3 & 6 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 6 & -2 \\ 3 & 5 & 1 \\ -3 & 4 & 0 \end{bmatrix}$

$(2\times3)(3\times3)$

2×3 matrix as the result

$= \begin{bmatrix} -1(0)+3(3)+6(-3) & -1(6)+3(5)+6(4) & -1(-2)+3(1)+6(0) \\ 4(0)+0(3)+2(-3) & 4(6)+0(5)+2(4) & 4(-2)+0(1)+2(0) \end{bmatrix}$

$= \begin{bmatrix} -9 & 33 & 5 \\ -6 & 32 & -8 \end{bmatrix}$

Properties of Matrix Multiplication Let A, B and C be matrices such that all of the matrix products below are defined and let r be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$ re-group
- **Associative Property with Scalar Multiplication:** $r(AB) = (rA)B = A(rB)$ + Commutativity
- **Identity Property:** For a natural number k, the $k \times k$ **identity matrix**, denoted I_k, is a square matrix containing 1's down the main diagonal and 0's elsewhere. For a $m \times n$ matrix A $\boxed{I_m A = AI_n = A}$.
- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$